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DETERMINATION OF CRITICAL NONDIMENSIONAL PARAMETERS IN AIRCRAFT DYNAMIC RESPONSE TO RANDOM INPUT

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16. ABSTRACT

The critical parameters of subsonic jet aircraft response in a random atmospheric environment are determined. Equations of motion are presented for semirigid aircraft with a flexible primary airfoil. However, the analysis is easily extendable to include additional appendage flexibility. The analysis establishes the mechanical admittance values for pitching, plunging, and the first mode effects from wing elastic bending and torsion. Nondimensional parameters are established which allow the representation of all subsonic jet transport aircraft with one nondimensional model. The critical parameters for random forcing are found to be aircraft relative mass, reduced natural and forcing frequencies, and Mach number. Turbulence scale lengths are found to be directly related to the critical values of reduced forcing frequency. Results are given for subsonic craft traveling at constant altitude. Specific values of admittance functions are tabulated at Mach numbers of 0.2, 0.5, and 0.7. The relative mass range covers all aircraft currently in operation.

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FOREWORD

This research was conducted by The Pennsylvania State University for the National Aeronautics and Space Administration, George C. Marshall Space Flight Center, Huntsville, Alabama. Dr. George H. Fichtl of the Aerospace Environment Division, Aero-Astrodynamics Laboratory, was the Scientific Monitor. Professor John A. Dutton, Department of Meteorology, The Pennsylvania State University, was the Principal Investigator. The support for this research was provided by Mr. John Enders of the Aeronautical Operating Systems Division, Office of Advanced Research and Technology, NASA Headquarters.

The research reported in this document was motivated by the need to indicate critical atmospheric parameters in relation to associated critical aircraft response characteristics for the safe design and operation of aeronautical systems. This research is part of an investigation of the effect of severe atmospheric forcing on aeronautical systems. The report is a Master of Science Thesis in Aerospace Engineering prepared by Mr. Hillard in collaboration with and under the direction of Professor Maurice M. Sevik, Professor of Aerospace Engineering, The Pennsylvania State University.

A previous report presented results of an empirical study of thunderstorm gust fronts, and a subsequent report will present results from a numerical model of gust front structure and evolution.

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NOMENCLATURE

$\bar{a}$	displacement of elastic axis from one half chord position
b	semi chord length
$\bar{b}$	ratio of wing chord to tail chord
$\bar{c}$	mean aerodynamic chord
$\hat{h}$	vertical plunging displacement coordinate
$\underline{h}$	angular momentum vector
h_x, h_y, h_z	aircraft angular momentum components
i_B	moment of inertia parameter
K	wing reduced frequency parameter
K_t	tail reduced frequency parameter
ℓ	semi span length
ℓ_t	tail semi span length
$\bar{m}$	wing mass distribution (nondimensional)
t	wing thickness or time
u_o	forward flight velocity
$\underline{V}_c$	center of mass velocity
w_G	gust velocity
A	aircraft characteristic area
B	spanwise moment of inertia
C_w	weight coefficient[24]
C_m	moment coefficient[24]
C_b	bending coefficient[24]
C_d	stress coefficient[24]
D	nondimensional differential operator
F_z	force in the vertical direction

E	Young's modulus
G	shear modulus
$H_j\ (i\omega)$	plunging and bending mechanical admittances
$T_j(i\omega)$	pitching and torsional mechanical admittance
I_x	wing section moment of inertia
I_o	wing mass moment of inertia
$\bar{I}_o$	wing mass moment of inertia (nondimensional)
J	wing polar moment of inertia
L^M	lift due to motion
L^G	lift due to gust encounter
$L_h, L_\alpha, M_h, M_\alpha$	aerodynamic coefficients
M^M	moment due to motion
M	pitching moment or weighted wing mass distribution
Q_h	generalized force
Q_α	generalized moment
$C_\alpha, C_\alpha', C_\alpha''$	torsional frequency coefficients
C_h, C_h', C_h''	bending frequency coefficients
$P,\ Q,\ R$	aircraft angular velocity components
$U,\ V,\ W$	aircraft linear velocity components
S	wing planform area
S_α	wing static moment
W	gross aircraft weight
ΔZ	unsteady lift
ΔM	unsteady moment

a_{ij}	elasticity integral $\int_0^{\ell} \frac{\phi_i \phi_j}{\ell} dy$
g_i	$\int_0^{\ell} \frac{\phi_i(y)}{\ell} dy$
S_{ij}	torsional integral $\int_0^{\ell} \frac{\psi_i \psi_j}{\ell} dy$
h_i	$\int_0^{\ell} \frac{\psi_i(y)}{\ell} dy$
W_S	aircraft structural weight
α	airfoil angle of attack
α_G^*	gust velocity
$\bar{\alpha}_G^{**}$	nondimensional gust velocity
β, L	turbulence length (wavelength)
ε	downwash angle
ε_o	zero lift downwash angle
$\bar{\phi}$	phase between wing and tail
$\phi_j(y)$	bending mode shape
$\psi_j(y)$	torsional mode shape
δ	Kronecker delta
$\Phi(K)$	Sear's function
$\lambda_1, \lambda_2, \lambda_3, \lambda_4$	distances from c.g. to unsteady lift locations
$\underline{\omega}$	angular velocity vector
ω	forcing frequency
ω_h	natural frequency of bending mode
ω_α	natural frequency of torsion mode
θ	rigid body pitching angle
μ	relative mass parameter

ρ	fluid density
ν	Poisson's ratio
σ_s	shear stress
σ_b	bending stress
$\bar{\lambda}$	turbulence spectra wavelength

1.0 INTRODUCTION

1.1 Previous Investigations

The local fluctuations of air velocity encountered by a craft moving through atmospheric turbulence are random functions of position and time which are computable only in a statistical sense. They, therefore, produce aerodynamic forces and moments which are likewise expressible only in a statistical sense. If the velocity field can be approximated by one for which the turbulence is statistically stationary and ergodic, mathematical techniques have been developed which may be used to find the rigid body and elastic response of the aircraft [1].

Several investigators have treated the problem of gust encounter. The first significant contributions were made by Wagner [2] 1925, who considered the problem of a step change in angle of attack. Kussner [3] 1932, considered lift generation from a sharp edge gust. A fundamental method was presented by von Karman [4] 1934 for the treatment of arbitrary airfoil motion. In his method, von Karman determined the system of impulses which are applied by the wing to the fluid. He accomplished this by investigating the three dimensional vortex wake phenomenon. Sears [5] 1940, introduced a Laplace transformation on the time variable which greatly simplified the numerical integration which was necessary in the use of von Karman's method. Sears also calculated a function which predicts the lift due to a sinusoidal gust. The general case of an inclined sinusoidal gust has been treated by Filotas [6] 1967 who obtained a closed form expression

for the loading on a two dimensional airfoil. Horlock [7] 1968 treated the component of gust which acts parallel to the wing chord and developed a function which yields the contribution to lift of this component. Naumann and Yeh [8] 1972 have added the effect of a cambered airfoil to Horlock's analysis. Sear's function has been used in conjunction with two dimensional lifting surface theory by Giesing [9] 1969 for the treatment of harmonic gusts.

The finite wing has been treated by several investigators. Jones [10] 1940 calculated the derivative coefficients for airfoils of finite span and also for airfoil-aileron-tab combinations. Reissner [11] 1943 studied the effect of finite span upon the airloads on oscillating wings for subsonic compressible flow. In an unpublished paper by Graham [12], he succeeds in obtaining a numerical solution for the load on a thin rectangular wing in an arbitrary incompressible flow field.

Random atmospheric forcing is generally treated using power spectral techniques. Diederich and Eggleston [13] 1956 calculated the power spectra of moments on a wing in random turbulence neglecting the effect of side gusts and spanwise variations in gust intensity. Diederich and Drischler [14,16] 1957 included spanwise gust variations in their investigation of sharp edged traveling gusts and the lift occurring from penetration of these gusts. Houbolt [15] 1970 has added important procedural methods for establishing gust load levels appropriate for use in aircraft structural design. A method of establishing load exceedance curves for structural fatigue analysis was presented in this study.

The analyses due to Diederich and Eggleston [12], and Diederich and Drischler [13], contain several fundamental assumptions. It is assumed that the encountered turbulence is homogeneous and isotropic. It is further assumed that the turbulence displays gust intensity invariance along the flight path until the craft has traversed a given region of turbulence. Two dimensional, unsteady aerodynamics for incompressible flow was used in these studies. Although the above assumptions are compatible to certain cases, their restrictions limit the generality of the results obtained.

Houbolt [15] 1970, used power spectral techniques to treat the problem of vertical gusts that are uniform in the spanwise direction. The atmospheric turbulence environment is modelled in terms of a spectrum shape. The parameters of this model are the r.m.s. gust intensity, turbulence scale length, and a parameter which allows for the flight time spent in the turbulence. The response is given in the frequency domain. Gust load levels are established for aircraft structural design. Houbolt's work provides systematic procedures for structural response to random conditions with no more difficulty than is usually associated with a discrete gust analysis.

1.2 Investigation of Severe Atmospheric Forcing

Severe atmopsheric phenomena affect several aspects of aircraft design. Adequate protection to systems must include the effects of severe conditions on returning space vehicles, vertical take-off aircraft, automatic control systems, and their respective structural integrities.

Statistical techniques now being utilized to investigate the response of a vehicle to atmospheric forcing make two basic assumptions: (a) the system is linear and (b) the input is realized from a stationary Gaussian process. These assumptions are not always true.

The investigation of atmospheric forcing necessarily includes the basic properties of the craft's response. The correlation of the response to the forcing function provide valuable insight in two important areas. First such analysis provides information to the systems designer, and secondly it provides the meteorologist with clues as to what turbulence parameters most strongly affect such systems. It is with these two factors in mind that this study is undertaken.

1.3 Statement of the Problem

Designing for unusual atmospheric conditions requires the determination of structural stress levels, rigid body motions, and elastic responses of the craft. The aerodynamic forces emanating from random forcing are comprised of a component due to the external gust input and a component due to the craft's rigid and elastic responses to such input. If these contributions are calculable, then the responses can be found.

The specific problem to be considered in this study is that of an aircraft in steady level flight encountering a spanwise gust wave, Figure 1.1. The previous investigations of Diederich and Eggleston [13], Diederich and Drischler [14], and Houbolt]15] have dealt with the correlation functions and power spectra of the forces and moments which occur to a craft in a turbulent environment. This study treats those

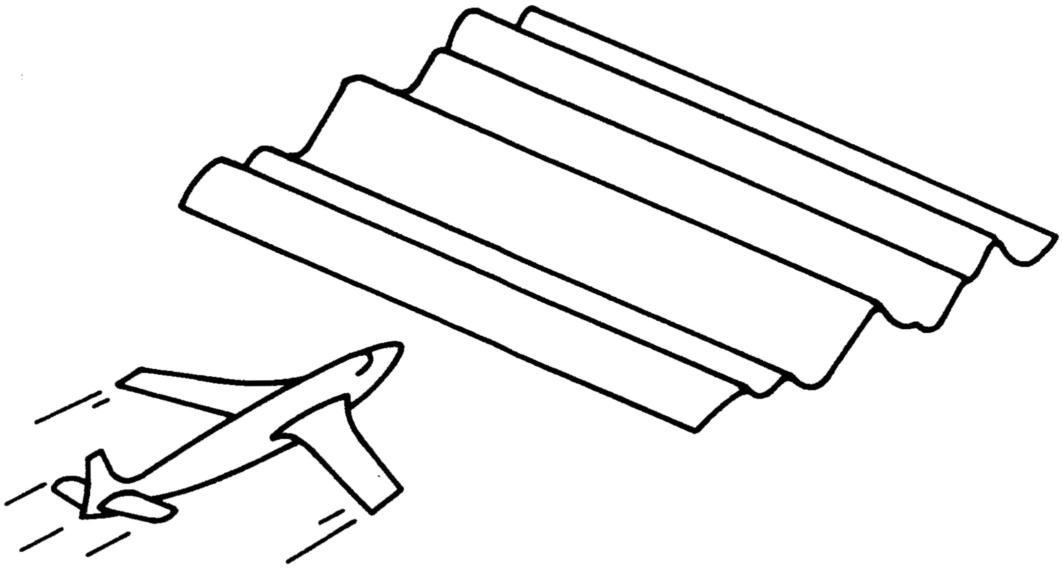

Figure 1.1 - Spanwise Gust Wave Encounter

aircraft parameters and meteorological parameters which are most critical to aircraft response. Results are presented which illustrate the dependence of these critical parameters with respect to Mach number, relative mass, and natural frequency. In order to reduce the mathematical complexity of the problem without compromising the significance of the results and conclusions, the craft's wing shall be unswept and untapered.

2.0 THE RIGID BODY EQUATIONS OF MOTION

2.1 Review of the Equations of Motion

The force and moment acting on and about the center of mass of an aircraft are directly derivable from Newton's second law. The center of mass has the linear velocity components U, V, and W, and angular momentum components h_x, h_y, and h_z. Their time derivatives are given by

$$(2.1) \qquad \frac{d\underline{V}c}{dt} = \frac{d\underline{V}c}{\delta t} + \omega \times Vc$$

$$(2.2) \qquad \frac{d\underline{h}}{dt} = \frac{\delta \underline{h}}{\delta t} + \underline{\omega} \times \underline{h}$$

where the coordinate system is attached to the craft and in motion with it. From expressions (2.1) and (2.2), the equations describing the plunging and pitching motions of an aircraft become:

$$(2.3) \qquad F_z = m\ (W + PV - QU)$$

$$(2.4) \qquad M = h_y + Rh_x - Ph_z$$

where P, Q, and R are the angular velocity components.

The forces acting on an aircraft are gravitational and aerodynamic in nature. The usual assumption may be made that the disturbances are small deviations from the reference flight state. This assumption gives a reasonable approximation for disturbances to θ less than $\pi/6$. Generally flight of considerable turbulence occurs with quite small linear and angular magnitudes. The variables may

be assumed to be the sum of their reference flight value and a small disturbance value, thus enabling the linearization of the equations of motion:

$$\Delta Z = m \frac{d^2\hat{h}}{dt^2} - mu_o \frac{d\theta}{dt} \tag{2.5}$$

$$\Delta M = B \frac{d^2\theta}{dt^2} \tag{2.6}$$

The equations may be expressed in nondimensional form by dividing (2.5) and (2.6) respectively by $1/2\ \rho u_o^2 S$ and $1/2\ \rho u_o^2 S\bar{c}$. The resulting expressions are given by,

$$\Delta\bar{Z} = \mu D\ (\bar{D}h - \theta) \tag{2.7}$$

$$\Delta\bar{M} = 2i_B D^2\theta \tag{2.8}$$

2.2 Relations of Unsteady Lift and Moment

The unsteady lift and moment forcing functions must be expressed with respect to a set of coordinate axes fixed in the aircraft as follows:

$$L^M = -\pi\rho b^3\omega^2 \{L_h \frac{\hat{h}}{b} - [L_\alpha - (\frac{1}{2} + \bar{a})\ L_h]\ \alpha\} \tag{2.9}$$

$$M_y^M = \pi\rho b^4\omega^2 \{[M_h - (\frac{1}{2} + \bar{a})\ L_h]\ \frac{\hat{h}}{b} - [M_\alpha - (\frac{1}{2} + \bar{a})\ (L_\alpha + M_n) + (\frac{1}{2} + \bar{a})^2\ L_n]\ \alpha\} \tag{2.10}$$

These relations then give the component of unsteady force and moment due to the aircraft's response. The lift generated on an airfoil due

to a sinusoidally varying gust may be expressed in terms of Sears function $\Phi(k)$; it is located at the quarter chord point. The lift per unit span is given by

$$L^G = 2\pi\rho u_o^2 b\alpha_G^* \Phi(k) e^{i\omega t} \tag{2.11}$$

In equations (2.9) and (2.10), the lift and moment are applied at and about the elastic axis. The aerodynamic coefficients and Sear's function appearing in these equations have the numerical values shown in Smilg and Wassermann.

In order to obtain the total force and moment acting on the craft, the effects of the above stated forces and moments must be summed over the wing and tail. The total force ΔZ and moment ΔM may be expressed as;

$$\Delta Z = 2\int_o^{\ell} [L_w^M + L_w^G + L_t^M + L_t^G]dy \tag{2.12}$$

$$\Delta M = 2\int_o^{\ell} [M_w + M_t - \hat{\lambda}_1 L_t^M + \hat{\lambda}_2 L_w^M - \hat{\lambda}_3 L_t^G + \hat{\lambda}_4 L_w^G]dy \tag{2.13}$$

The subscripts on the moment and lift terms indicate the wing or tail contribution. The superscript indicates the source of the term to be either gust or motion. The terms $\hat{\lambda}_i$ indicate geometrical distances from the center of gravity to the points of application of the forces. These distances are shown in Figure 2.1.

Substituting equations (2.12) and (2.13) in nondimensional form into equations (2.7) and (2.8), the rigid body equations of motion are obtained. In Chapter 4.0 of this study it will be shown that the tail frequency parameter k_t is equal to 0.6k for a large class of subsonic aircraft. At this time the relation will be used without proof.

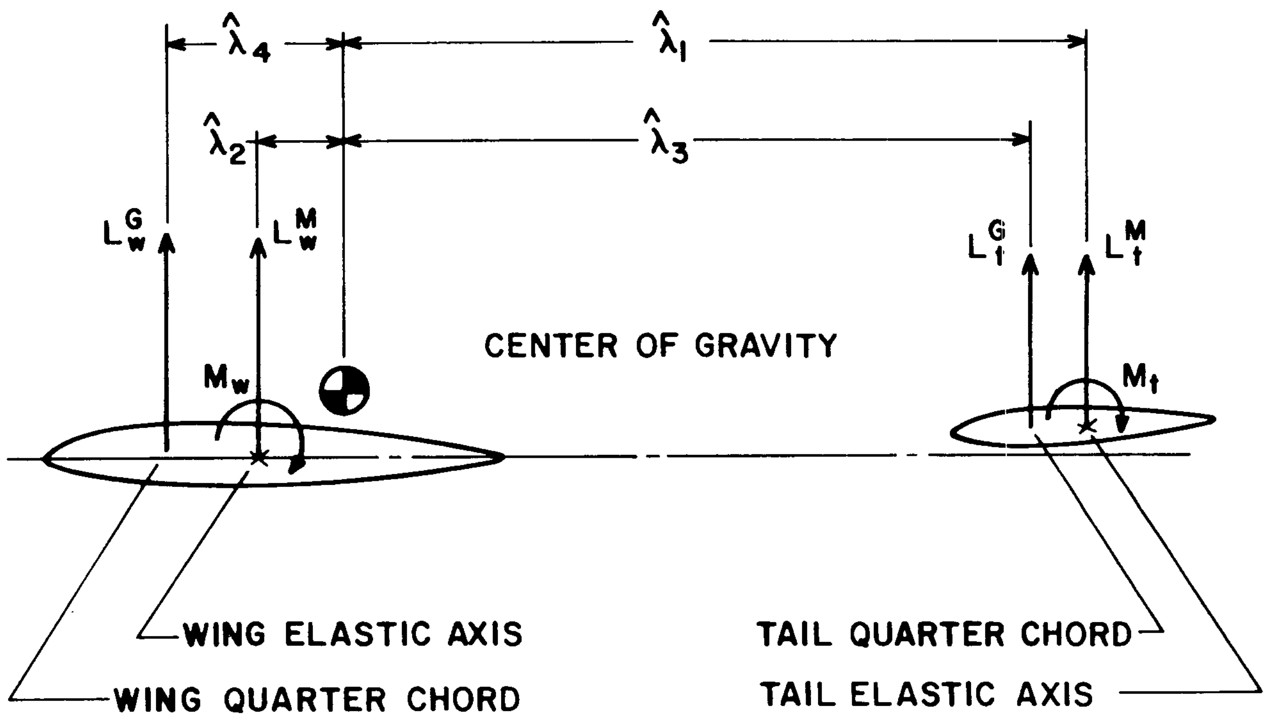

Figure 2.1 - Location of Geometrical Distances

Note should also be made that at any given instant of time, the tail forces will have a phase difference with respect to the wing forces. A simplification to this situation will also be given in Section 2.6.

2.3 Solutions for the Unknown Normal Coordinates

The general form of the solution of the equations of motion may be expressed as the sum of the products of the mode shapes with corresponding generalized coordinates as follows.

$$\hat{h} = \sum_j \phi_j(y)\xi_j(t) \tag{2.14}$$

Since ξ_j varies sinusoidally with time, it may be expressed in terms of the mechanical admittance of the system as follows:

$$\xi_j(t) = H_j(i\omega)e^{i\omega t} \tag{2.15}$$

Therefore, equation (2.14) becomes

$$\hat{h} = \sum_j \phi_j(y)H_j(i\omega)e^{i\omega t} \tag{2.16}$$

The mechanical admittance functions occurring in the rigid body equations of motion contains four terms, namely: plunging $H_1(i\omega)$, elastic bending $H_2(i\omega)$, pitching $T_1(i\omega)$, and torsion $T_2(i\omega)$.

The coordinates pertinent to these equations and their derivatives may then be expressed as

$$(2.17) \qquad \hat{h}_w = \sum_j \phi_j(y) H_j(i\omega) e^{i\omega t}$$

$$(2.18) \qquad \hat{h}_t = \sum_j \phi_j(y) H_j(i\omega) \delta_{ij} e^{i\omega t} + \lambda_3 \sum_j \psi_j(y) T_j(y) T_j(i\omega) \delta_{1j} e^{i\omega t}$$

$$(2.19) \qquad \alpha_w = \left(\frac{w_G}{u_o}\right) + \frac{(i\omega)}{u_o} \sum_j \phi_j(y) H_j(i\omega) e^{i\omega t}$$

$$+ \lambda_2 \frac{(i\omega)}{u_o} \sum_j \psi_j(y) T_j(i\omega) e^{i\omega t}$$

$$(2.20) \qquad \alpha_t = \left(\frac{w_G^*}{u_o}\right) + \frac{(i\omega)}{u_o} \sum_j \phi_j(y) H_j(i\omega) \delta_{1j} e^{i\omega t}$$

$$+ \lambda_3 \frac{(i\omega)}{u_o} \sum_j \phi_j(y) T_j(i\omega) \delta_{1j} e^{i\omega t}$$

$$(2.21) \qquad \theta = \sum_j \psi_j(y) T_j(i\omega) \delta_{1j} e^{i\omega t}$$

$$(2.22) \qquad \ddot{\theta} = (i\underline{\omega})^2 \sum_j \psi_j(y) T_j(i\omega) \delta_{1j} e^{i\omega t}$$

The term θ represents the rigid body angular displacement coordinate. The term α represents the total angle of attack. This includes in addition to all response coordinates, the effect of the vertical gust and plunging velocity on the angle of attack. The term δ_{1j} equals 1 when $j=1$ and zero otherwise and signifies that only rigid modes are included. It should also be noted that w_g^* includes the phase difference which the gust velocity at the tail has with respect to that acting on the wing. While the equations themselves describe the rigid body behavior of the craft, the forcing functions contain elasticity and torsional contributions to this behavior.

2.4 Rigid Body Equations of Motion

The equations of motion for the rigid body response of the aircraft may now be stated. The following substitutions for terms containing aerodynamic coefficients are made:

$$L_1(k) = [L_\alpha - (\tfrac{1}{2} + \bar{a})\, L_h] \tag{2.23}$$

$$L_2(k) = [M_h - (\tfrac{1}{2} + \bar{a})\, L_h] \tag{2.24}$$

$$L_3(k) = [M_\alpha - (\tfrac{1}{2} + \bar{a})\,(L_\alpha + M_h) + (\tfrac{1}{2} + \bar{a})^2\, L_h] \tag{2.25}$$

$$L_4(k_t) = [L_\alpha - (\tfrac{1}{2} + \bar{a}_t)\, L_h]_t \tag{2.26}$$

$$L_5(k_t) = [M_h - (\tfrac{1}{2} + \bar{a}_t)\, L_h]_t \tag{2.27}$$

$$L_6(k_t) = [M_\alpha - (\tfrac{1}{2} + \bar{a}_t)\,(L_\alpha + M_h) + (\tfrac{1}{2} + \bar{a}_t)^2\, L_h]_t \tag{2.28}$$

The equation for rigid body plunging is given by

$$\begin{aligned} & H_1(ik)\,\{4(\underline{ik})^2\,[\mu - \pi(L_h + (\underline{ik})\,L_1)\int_0^{\ell} \frac{\phi_1(y)}{\ell}\,dy \\ & \quad - \pi\bar{b}^2\,(L_{h_t} + (\underline{ik})\,L_4)\int_0^{\ell_t} \frac{\phi_1(y)}{\ell}\,dy]\} \\ & + H_2(ik)\,\{-4\pi\,(\underline{ik})^2\,(L_h + (\underline{ik})\,L_1)\int_0^{\ell} \frac{\phi_2(y)}{\ell}\,dy\} \\ & + T_1(ik)\,\{-2(\underline{ik})\,[\mu + 2\pi\,(\underline{ik})^2\,\hat{\lambda}_3 L_1\int_0^{\ell} \frac{\psi_1(y)}{\ell}\,dy \\ & + 2\pi\bar{b}^2(\underline{ik})\,\hat{\lambda}_3\,(L_{h_t} + \bar{b}(\underline{ik})\,L_4)\int_0^{\ell_t} \frac{\psi_1(y)}{\ell}\,dy]\} \\ & + T_2(\dot{ik})\,\{4\pi(\underline{ik})^3\,\hat{\lambda}_2 L_1\int_0^{\ell} \frac{\psi_2(y)}{\ell}\,dy\} \end{aligned} \tag{2.29}$$

$$= 2\pi \left(\frac{w_G}{u_o}\right) \{[(\underline{ik})^2 L_1 + 2\Phi(k)] \int_o^{\ell} \frac{dy}{\ell}$$

$$+ \bar{b}e^{i\phi} [\bar{b}^2(\underline{ik})^2 L_4 + 2\Phi(k_t)] \int^{\ell_t} \frac{dy}{\ell}\}$$

Similarly the equation for rigid body pitching motion may be written as follows:

(2.30)
$$H_1(ik) \{2\pi(\underline{ik})^2 [L_2 + (\underline{ik}) L_3 + 2\hat{\lambda}_2 L_h$$

$$+ 2(\underline{ik}) \hat{\lambda}_2 L_1] \int_o^{\ell} \frac{\phi_1(y)}{\ell} dy$$

$$+ 2\pi(\underline{ik})^2 \bar{b}^2 [b L_5 + \bar{b}^2 (\underline{ik}) L_6 + 2\hat{\lambda}_3 L_{h_t}$$

$$+ 2(\underline{ik}) \bar{b} \hat{\lambda}_3 L_4] \int_o^{\ell_t} \frac{\phi_1(y)}{\ell} dy\}$$

$$+ H_2(ik) \{2\pi(\underline{ik})^2 [L_2 + (\underline{ik}) L_3 + 2\hat{\lambda}_2 L_h + 2(\underline{ik}) \hat{\lambda}_2 L_1]$$

$$\cdot \int_o^{\ell} \frac{\phi_2(y)}{\ell} dy\}$$

$$+ T_1(ik) \{2(\underline{ik})^2 [4i_B + \pi(ik) \{(\hat{\lambda}_2 L_3 + 2\hat{\lambda}_2^2 L_1) \int_o^{\ell} \frac{\psi_1(y)}{\ell} dy$$

$$+ (\bar{b}^4 \hat{\lambda}_3 L_6 + 2\bar{b}^3 \hat{\lambda}_3^2 L_4) \int_o^{\ell_t} \frac{\psi_1(y)}{\ell} dy]\}$$

$$+ T_2(ik) \{2\pi(\underline{ik})^3 \hat{\lambda}_2 (L_3 + 2\hat{\lambda}_2 L_1) \int_o^{\ell} \frac{\psi_2(y)}{\ell} dy\} =$$

$$- \pi\left(\frac{w_G}{u_o}\right)e^{i\phi} [4\hat{\lambda}_1 \bar{b}\Phi(k_t) + \bar{b}^4(\underline{ik})^2 L_6 + 2\bar{b}^3(\underline{ik})^2 \hat{\lambda}_3 L_1] \int_o^{\ell_t} \frac{dy}{\ell}$$

$$-\pi(\underline{ik})^2 \left(\frac{w_G}{u_o}\right) [L_3 + 2\hat{\lambda}_2 L_1] \int_o^{\ell} \frac{dy}{\ell}$$

The equations as derived provide two relations necessary for computation of the longitudinal rigid response of a craft. The inclusion of bending and torsion modes necessitate the introduction of additional equations, the number of which will correspond to the number of elastic modes considered.

2.5 Effect of Downwash

The aerodynamic characteristics of the component parts of an aircraft are affected by each other. These interferences occur because of interactions between such components as the wing and fuselage, tail and fuselage, or the wing and the tail.

The most important effect is the downwash contribution that is due to the unsteady forces acting on the wing of the aircraft. Since the unsteady lift and moment expressions, (2.8) and (2.9), are dependent upon the instantaneous angle of attack of the tail, it is necessary to decrease this angle by the amount of downwash angle ε.

A general expression may be written for the unsteady moment acting on the craft, namely

$$(2.31) \qquad M_u = (\lambda_4 L^G - \lambda_2 L^M) \cos \alpha_w - (\lambda_1 L_t^G + \lambda_3 L_t^M) \cos (\alpha_w - \varepsilon) + M_w + M_t$$

The angles α_w and $(\alpha_w - \varepsilon)$ may be assumed small. Equation (2.31) may therefore be written in nondimensional form as

$$(2.32) \qquad C_{M_u} = (\hat{\lambda}_4 C_L^G - \hat{\lambda}_2 C_L^M - \hat{\lambda}_1 C_{L_t}^G - \hat{\lambda}_3 \, C_{L_t}^M + C_M = C_{M_t})$$

The terms of equation (2.32) which must be investigated are $\hat{\lambda}_3 C^M_{L_t}$ and C_{M_t}, and the significance of the downwash must be determined.

The time dependent downwash angle ε may be approximated by the expression,

$$(2.33) \qquad \varepsilon = \frac{\partial \varepsilon}{\partial \alpha}\alpha$$

We substitute in equation (2.32) the expressions for lift and moment, namely equations (2.9), (2.10), and (2.11) and obtain the following expression for the unsteady moment coefficient:

$$(2.34) \qquad C_{M_h} = H_1(ik)\,\{\pi k^2\,[2L_h\hat{\lambda}_2 - 2(\underline{ik})\,L_1 + L_2$$

$$- \tfrac{1}{2}(\underline{ik})\,L_3]\int_0^{\ell}\frac{\phi_1(y)}{\ell}\,dy + \pi k^2\,[2\bar{b}\hat{\lambda}_3\,L_{h_t} + \bar{b}^3\,L_5$$

$$+ (\underline{ik})\bar{b}^4\,(1 - \frac{\partial\varepsilon}{\partial\alpha})\,(L_4 + \frac{1}{2}\,L_6)]\int_0^{\ell_t}\frac{\phi_1(y)}{\ell}\,dy\}$$

$$+ H_2(ik)\,\{\pi k^2\,[(2\hat{\lambda}_2 L_h + L_2) - (\underline{ik})(2L_1 + \frac{1}{2}\,L_3)\int_0^{\ell}\frac{\phi_2(y)}{\ell}\,dy\}$$

$$+ T_1(ik)\,\{-\pi k^2(\underline{ik})\,\hat{\lambda}_2\,(2L_1 + \frac{1}{2}\,L_3)\int_0^{\ell}\frac{\psi_1(y)}{\ell}\,dy$$

$$+ \bar{b}^2\hat{\lambda}_3 k^2\,[(2\hat{\lambda}_3 L_{h_t} + \bar{b}L_5) + (\underline{ik})\,\bar{b}\,(L_4 + \frac{1}{2}\,L_6)\,(1 - \frac{\partial\varepsilon}{\partial\alpha})]$$

$$\int_0^{\ell_t}\frac{\psi_1(y)}{\ell}\,dy\}$$

$$+ T_2(ik)\,\{-\pi k^2(\underline{ik})\hat{\lambda}_2\,(2L_1 + \frac{1}{2}\,L_3)\int_0^{\ell}\frac{\psi_2(y)}{\ell}\,dy\}$$

$$+ \{\pi[-k^2\,(\frac{w_G}{u_o})\,L_1 - \frac{1}{2}\,k^2 L_3\,(\frac{w_G}{u_o}) + 2\hat{\lambda}_4 w_G\,\Phi(k)]\int_0^{\ell}\frac{dy}{\ell}$$

$$+ [\pi k^2 \bar{b}^{-3} L_4 (\frac{w_G}{u_o}) + \frac{\pi}{2} \bar{b}^4 k^2 (\frac{w_G}{u_o}) L_6] e^{i\bar{\phi}} (1 - \frac{\partial\varepsilon}{\partial\alpha}) \int_o^{\ell_t} \frac{dy}{\ell}$$

$$- 2\pi\hat{\lambda}_1 \bar{b} w_B \Phi(k_t) e^{i\phi} \int_o^{\ell_t} \frac{dy}{\ell} \}$$

This equation shows that the downwash effects appear only in the rigid body modes. A conservative estimate of the effect of downwash may be established by considering the relative change to equation (2.34) upon deletion of $\partial\varepsilon/\partial\alpha$. The contribution due to downwash effects will be of the order of less than ten percent for the unsteady moment. For conventional aircraft, a typical value of $\partial\varepsilon/\partial\alpha$ is 0.40.

Similarly for the downwash contribution due to lift, the lift on the aircraft may be expressed by:

$$(2.35) \qquad L_h = L_w^G + L_t^G + L_w^M + L_t^M$$

which in nondimensional form becomes equal to

$$(2.36) \qquad C_{L_h} = H_1(ik) \{\pi k^2 [(2L_h - (ik)L_1) \int_o^{\ell} \frac{\phi_1(y)}{\ell} dy$$

$$+ (2\bar{b}^2 L_{h_t} + \bar{b}^3 (\underline{ik}) L_4 (1 - \partial\varepsilon/\partial\alpha)) \int_o^{\ell_t} \frac{\phi_1(y)}{\ell} dy]\}$$

$$+ H_2(ik) \{\pi k^2 [2L_h - (\underline{ik}) L_1] \int_o^{\ell} \frac{\phi_2(y)}{\ell} dy\}$$

$$+ T_1(ik) \{-\pi k^2 [(\underline{ik}) \hat{\lambda}_2 L_1 \int_o^{\ell} \frac{\psi_1(y)}{\ell} dy + (\bar{b}^3 (\underline{ik}) L_4 \hat{\lambda}_3$$

$$+ 2\bar{b}^2 \lambda_3 L_{h_t}) (1 - \frac{\partial\varepsilon}{\partial\alpha}) \int_o^{\ell_t} \frac{\psi_1(y)}{\ell} dy\}$$

$$+ T_2(ik)\ \{-\pi k^2(ik)\hat{\lambda}_2 L_1 \int_0^{\ell} \frac{\psi_2(y)}{\ell}\,dy\} =$$

$$\{\pi\bar{b}e^{i\phi}\ [2w_G\Phi(k_t) + k^2\bar{b}^2 L_4\ (\frac{w_G}{u_o})\ (1 - \frac{\partial\varepsilon}{\partial\alpha})] \int_0^{\ell_t} \frac{dy}{\ell}$$

$$+ \pi[2w_G\Phi(k) - k^2\ (\frac{w_G}{u_o})\ L_1] \int_0^{\ell} \frac{dy}{\ell}\}$$

Evaluation of equation (2.36) for the effect of downwash yields a contribution of not more than twelve percent of the unsteady lift coefficient. It may, therefore, be concluded that the effects of downwash may be safely neglected in this analysis.

2.6 Wing to Tail Phase Difference

Due to their spatial separation a phase difference $\bar{\phi}$ will occur between the wing and tail. For a given atmospheric disturbance of wavelength β, the time required to traverse a cycle is β/u_o. Similarly, the distance from wing to tail is approximately λ_3 and the time required to traverse this distance λ_3/u_o. In terms of the frequency, the phase difference may be expressed as

$$(2.37)\qquad \bar{\phi} = \frac{\omega\lambda_3}{u_o}$$

which may be written in the form,

$$(2.38)\qquad \bar{\phi} = 2k\hat{\lambda}_3$$

As is shown in Chapter 4.0, the nondimensional distance $\hat{\lambda}_3$ is a constant over the entire range of aircraft considered. The reduced frequency k therefore dictates the value of the phase difference.

3.0 ELASTIC EQUATIONS OF MOTION

3.1 Introduction

An aircraft is a single elastic body. The interactions of the elastic components present many complications in the analysis of such a structure. It is therefore analytically convenient to consider a craft as either rigid or semi rigid. In many cases this procedure will yield a reasonable engineering approximation.

For this analysis the craft will be assumed to be rigid except for the inclusion of wing elastic bending and torsion in their fundamental modes.

3.2 Lagrange's Formulation of the Equations of Motion for an Elastic Wing

Assuming the actual motion of a wing to be comprised of a combination of fundamental wing bending and wing torsion, the system may be expressed as an equivalent system consisting of an airfoil section of unit span restrained by springs against bending and torsional motion [17].

Using the Lagrangian approach, the equations of motion are given by

$$M\ddot{h} + S_\alpha \ddot{\alpha} + M\omega_h^2 h = Q_h \tag{3.1}$$

$$S_\alpha \ddot{h} + I_\alpha \ddot{\alpha} + I_\alpha \omega_\alpha^2 \alpha = Q_\alpha \tag{3.2}$$

where the generalized force and moment Q_h and Q_α represent the unsteady forcing to the wing per unit span. The generalized forcing terms may

be expressed using the quantities which have been previously defined in Chapter 2.0 and are restated here for convenience:

$$(2.9) \qquad L^M = -\pi\rho b^3\omega^2 \{L_h \frac{\hat{h}}{b} - [L_\alpha - (\frac{1}{2} + \bar{a})\, L_h]\, \alpha\}$$

$$(2.10) \qquad M_y^M = \pi\rho b^4\omega^2 \{[M_h - (\frac{1}{2} + \bar{a}\_\, L_h] \frac{\hat{h}}{b} - [M_\alpha - (\frac{1}{2} = \bar{a})\, (L_\alpha + M_n)$$

$$+ (\frac{1}{2} + \bar{a}\_^2\, L_n]\, \alpha\}$$

$$(2.11) \qquad L^G = 2\pi\rho u_o^2 b\alpha_G^* \Phi(K) e^{i\omega t}$$

The generalized force and moment are, therefore, of the form

$$(3.3) \qquad Q_h = L_w^G + L_w^M$$

$$(3.4) \qquad Q_\alpha = L_w^G\, e_1 + M_y$$

These equations may be nondimensionalized in a manner analogous to the rigid body equations of Chapter 3.0. One unique distinction should be made, however. The left hand side of equations (3.3) and (3.4) describe only the elastic bending and torsional modes and does not contain any rigid body considerations. The rigid body mode has previously been designated as the first mode thus necessitating the introduction of a Kroenecker delta which was written as $\bar{\delta}_{ij}$. For the elastic modes, the subscript i ranges from 2 to n, depending upon the number of modes considered.

3.3 Solution for the Equations

The elastic deformations are given by the sum of the products of the mode shapes and the corresponding normal coordinates as follows:

(3.5) $$h_w = \sum_j \phi_j(y)\, H_j(i\omega) e^{i\omega t}$$

(3.6) $$\alpha_w = \left(\frac{w_G}{u_o}\right) + \frac{(i\omega)}{u_o} \sum_j \phi_j(y)\, H_j(i\omega) e^{i\omega t} + \lambda_2 \frac{(i\omega)}{u_o} \sum_j \psi_j(y)\, T_j(i\omega) e^{i\omega t}$$

(3.7) $$h_2 = \sum_j \phi_j(y)\, H_j(i\omega) \bar{\delta}_{i_j}\, e^{i\omega t}$$

(3.8) $$\ddot{h}_2 = (\underline{i\omega})^2 \sum_j \phi_j(y)\, H_j(i\omega) \bar{\delta}_{i_j}\, e^{i\omega t}$$

(3.9) $$\alpha_2 = \sum_j \psi_j(y)\, T_j(i\omega) \bar{\delta}_{i_j}\, e^{i\omega t}$$

(3.10) $$\ddot{\alpha}_2 = (\underline{i\omega})^2 \sum_j \psi_j(y)\, T_j(i\omega) \bar{\delta}_{i_j}\, e^{i\omega t}$$

where h_w is the wing plunging coordinate and h_2 is the translational coordinate due to wing bending. The α_w term is the wing angle of attack, and α_2 the contribution to angle of attack due to wing torsional motion. The bending and torsional mode shapes are represented as $\phi_j(y)$ and $\psi_j(y)$. $H_j(i\omega)$ and $T_j(i\omega)$ are the admittance functions for the translational modes and torsional modes respectively.

These equations must next be integrated over the semi span length. It is advantageous - prior to the integration - to multiply the equations by $\phi_i(y)$ for the bending equation and $\psi_i(y)$ for the torsional equation. The orthogonality of the modes then yields:

(3.11) $$\int_o^{\ell} \bar{m}\, \phi_j(y)\, \phi_i(y)\, dy = \bar{M}\delta_{i_j}$$

$$(3.12) \qquad \int_o^{\ell} \bar{I}\, \psi_j(y)\, \psi_i(y)\, dy = \bar{\bar{I}}\delta_{i_j}$$

where δ_{ij} has its usual definition.

3.4 Equation for Elastic Bending

Substituting equations (3.3) through (3.10) into (3.1) and (3.2), we obtain the equation for elastic bending:

$$(3.13) \qquad H_1(ik)\,\{-2\pi(\underline{ik})^2\, a_{ij}\, [L_h + (ik)L_1]\}$$

$$+ H_2(ik)\,\{2a_{ij}\, [4(\underline{ik})^2\, \bar{m} - 4(\underline{ik})_h)^2\, \bar{m}$$

$$-\pi(\underline{ik})^2\, L_h - \pi(\underline{ik})^3 L_1]\}$$

$$+ T_1(ik)\,\{-2\, (\underline{ik})^3\, \hat{\lambda}_2 L_1 b_{ij}\}$$

$$+ T_2(ik)\,\{2(\underline{ik})^2\, b_{ij}\, [4\bar{S} - \pi(\underline{ik})\, \hat{\lambda}_2 L_1]\}$$

$$= \pi\, \{(\frac{w_G}{u_o})\, g_i\, [(\underline{ik})^2 L_1 + 2\Phi(k)]\}$$

3.5 Equation for Wing Torsion

Similarly the equation of torsion may be expressed as

$$(3.14) \qquad H_1(ik)\,\{\pi(\underline{ik})^2 s_{ij}\, [L_2 + (\underline{ik})L_3]\}$$

$$+ H_2(ik)\,\{s_{ij}(\underline{ik})^2\, [8\bar{S}_\alpha + \pi L_2 + \pi(\underline{ik})L_3]\}$$

$$+ T_1(ik)\,\{\pi(\underline{ik})^3\, \hat{\lambda}_2 L_3\, t_{ij}\}$$

$$+ T_2(ik) \{t_{ij} [8 (\underline{ik})^2 \bar{I}_\alpha - 8(\underline{ik}_\alpha)^2 \bar{I}_\alpha + \pi(\underline{ik})^3 \hat{\lambda}_2 L_3]\}$$

$$= \pi \left(\frac{w_G}{u_o}\right) h_i [2\hat{\lambda}_2 \Phi(k) - 1/2 (\underline{ik})^2 L_3]$$

The form of equations (3.13) and (3.14) indicates that they comprise a set of equations, the number of which corresponds to the number of modes which are to be included. The set of equations established now enables the solution of the functions $\bar{H}_j(ik)$ and $T_j(ik)$, for any desired number of elastic and torsional modes.

4.0 AIRCRAFT PARAMETER CHARACTERISTICS

4.1 Introduction

Examination of the equations of motion indicate that there exists a large number of variables. It would be computationally advantageous to reduce the number of parameters to a minimum. One would then be able to study how the various dimensionless parameters relate to each other for a large set of subsonic aircraft.

The equations of motion derived for an aircraft in Chapters 2.0 and 3.0 show dependence on aerodynamic coefficients and also on parameters which are directly attributable to the mass and geometrical characteristics of the craft. General observation of a particular class of aircraft indicate that they display similar basic characteristics. These similarities emanate from the function of the craft and the aerodynamic, control, and structural qualities which are necessary to accomplish this function. Since the parameters of a craft are relatively closely dependent upon its function, with the primary difference being the overall size of various craft within the class, one suspects that the various parameters might display some trends when considered over the entire range of that type of aircraft. Obviously the parameters should be plotted against a value indicating size. Gross take-off weight has been closen as most convenient.

The utility of displaying such characteristics derives from the ability of a designer to interpolate or extrapolate the trend to obtain a close approximation of the value for any craft desired. This can then be used to obtain useful information for the design of new aircraft or as is used in this study, the computation of the response of the craft to some input condition.

4.2 Classes of Parameters

As an example, the class of subsonic transport jets shall be considered. In general, the parameters displayed in the equations of motion fall into the following three distinct categories:

(a) Parameters which are dependent upon mass or mass distribution of the craft.

(b) Parameters which are dependent upon a geometrical length of the aircraft, but are not a function of spanwise location.

(c) Parameters which are a nondimensional geometrical length of the aircraft, but are a function of spanwise location.

In the analysis as presented in this study, the airfoils were assumed to be unswept and untapered thus eliminating the third type of parameter. It should be noted, however, that an extensive analysis of such craft will contain this type of parameter. A discussion of such parameters appears in Section 5 of this chapter.

The class of parameters dependent on mass and mass distribution contain μ the relative mass parameter, i_B the mass moment of inertia parameter, $\bar{S}$ which is the wing static moment term, $\bar{m}$ the nondimensional wing mass distribution parameter, and $\bar{I}$ the wing mass moment of inertia parameter. The nondimensional wing mass distribution terms are derived from the inclusion of elastic bending and torsional effects in the analysis. It is found that these quantities, due to their overall mass dependence, may be expressed as linear function of the relative mass parameter μ. It is, therefore, necessary only to specify μ in order that these quantities be defined.

The second class of parameters contain lengths or ratios of lengths in nondimensional form. These quantities are $\bar{b}$, the ratio of tail to wing mean aerodynamic chord, and $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_3$, and $\hat{\lambda}_4$ which are the nondimensional distances from the aircraft's center of gravity to the respective locations on the wing and tail where the unsteady aerodynamic forcing occurs.

As previously indicated, parameters of the third class have not been used in this analysis with the exception of the mode shapes $\phi_j(y)$ and $\psi_j(y)$ which occur because of elasticity considerations.

4.3 Parameters of the First Type

As previously mentioned, parameters of the first type may be defined by the specification of the relative mass parameter μ, and the mass moment of inertia parameter about the spanwise axis i_B, both representing nondimensional quantities.

The relative mass parameter is defined by the relation

$$\mu = \frac{W}{1/2 \ \rho g \ S\bar{c}} \tag{4.1}$$

The mass moment of inertia parameter is similarly defined as

$$i_B = \frac{B}{\rho \ S\bar{c}^3} \tag{4.2}$$

When the parameters defined by equations (4.1) and (4.2) are plotted against gross weight, the results are shown in Figures 4.1 through 4.5. Table 4.1 lists the crafts and their respective data which were used for parameter determination. Also shown in Figure 4.1 are the values of those wing parameters which can be seen to be linear functions of the relative mass parameter.

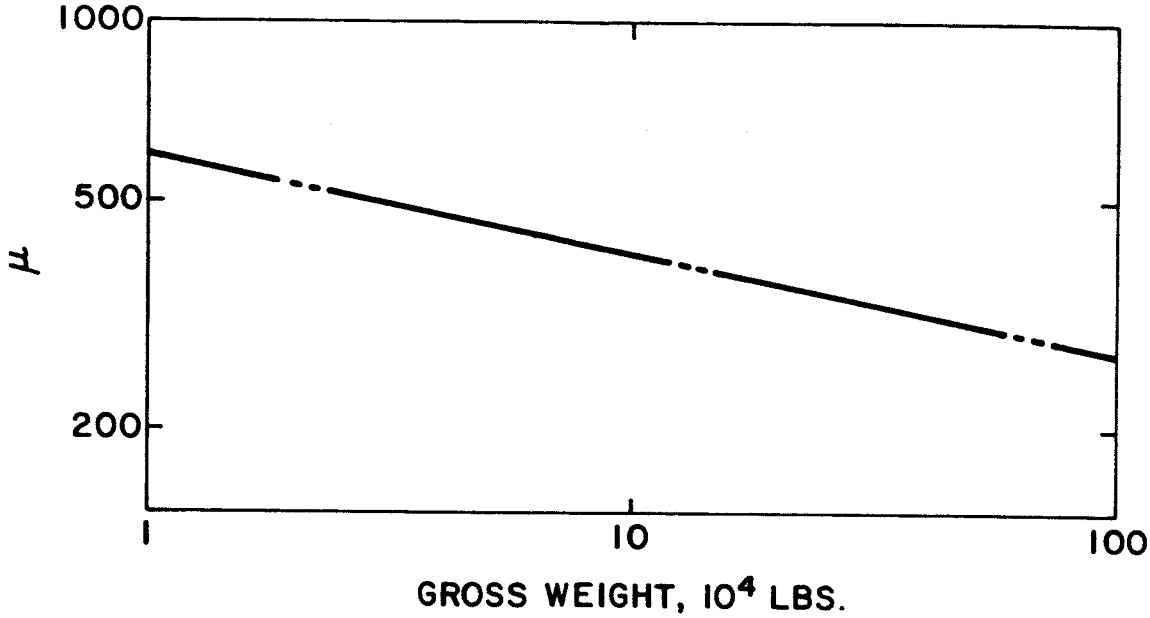

Figure 4.1 - Relative Mass Parameter

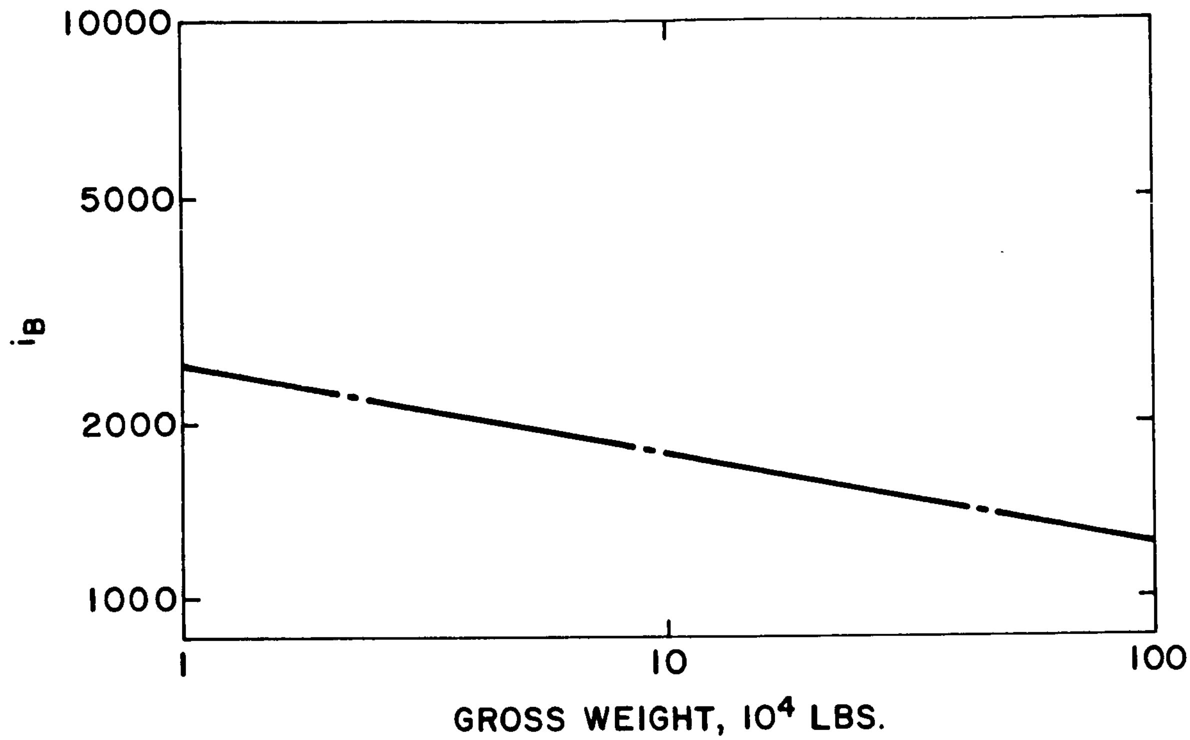

Figure 4.2 - Mass Moment of Inertia Parameter

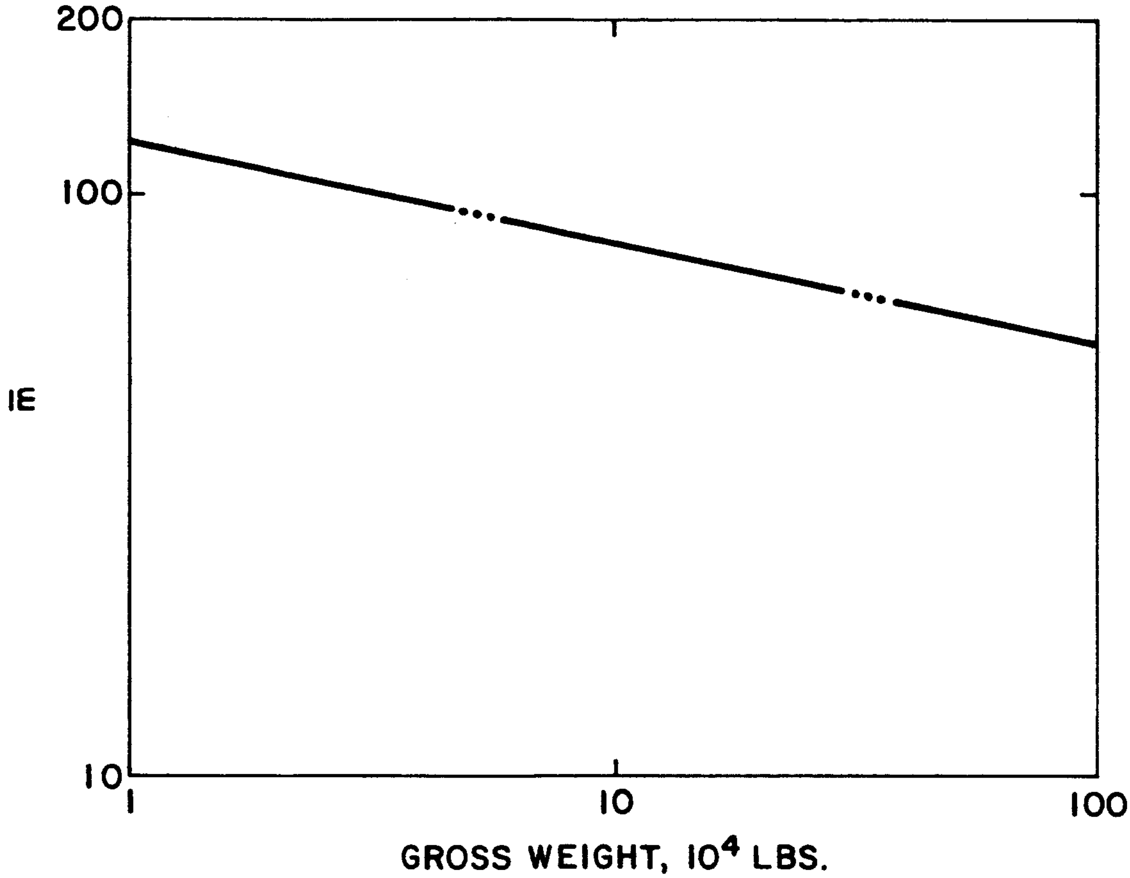

Figure 4.3 - Nondimensional Wing Mass Distribution

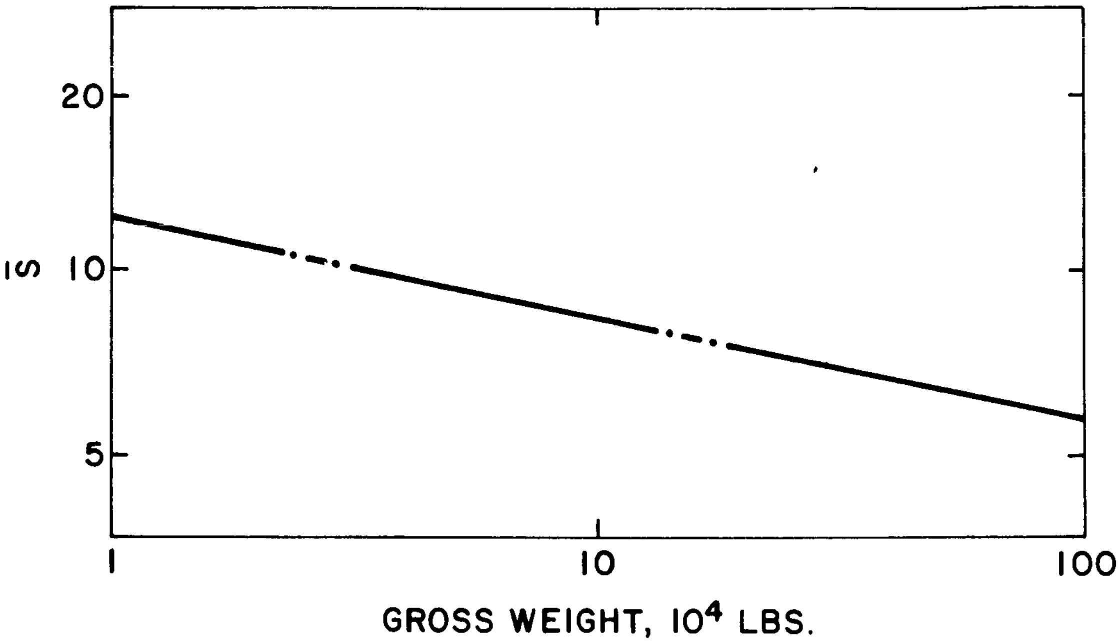

Figure 4.4 - Wing Static Moment Parameter

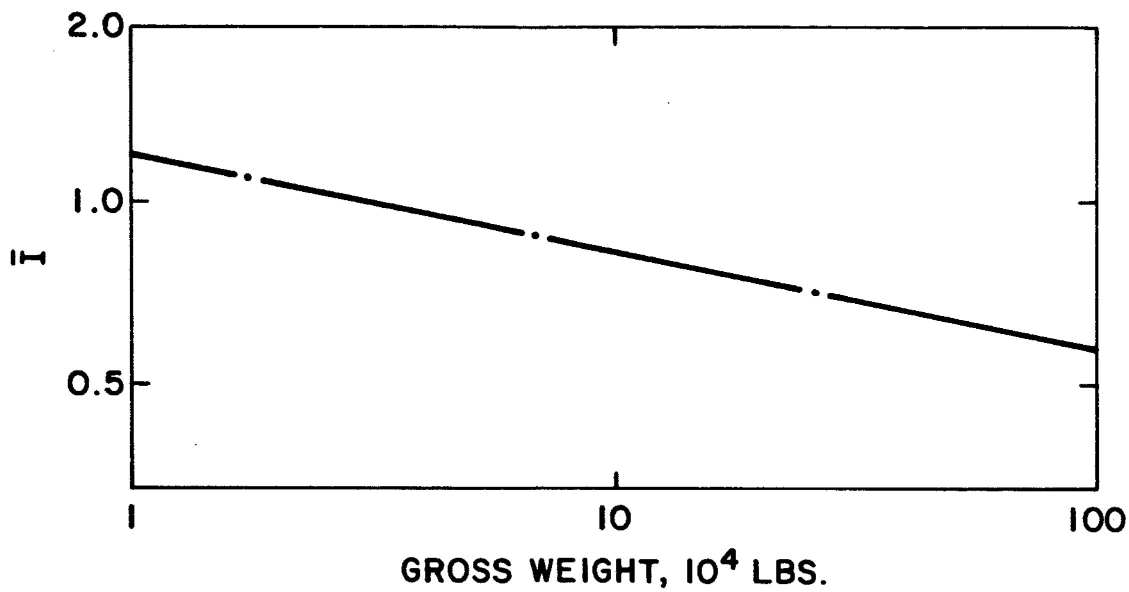

Figure 4.5 - Wing Mass Moment of Inertia

Table 4.1 Pertinent Aircraft Data

	Aircraft Designation	T.O. Weight
1.	Lear Jet 24	13,000
2.	Aero Jet Commander	16,800
3.	Dassault Fan Jet Falcon	26,455
4.	Lockheed Jet Star	40,921
5.	BAC One Eleven-200	78,500
6.	Boeing 737-200	93,500
7.	TU 134A	97,000
8.	Sud Aviation-210 Car.	101,413
9.	Hawker-Siddeley Trident	115,000
10.	Boeing 727-200	160,000
11.	Boeing 720	229,000
12.	Boeing 707-120B	257,000
13.	Douglas DC-8-10	273,000
14.	BAC VC10-1106	312,000
15.	Douglas DC-8-50	315,000
16.	Lockheed Starlifter	316,600
17.	Boeing 707-320B	328,000
18.	Boeing 747	680,000
19.	Lockheed C-5A	728,000

Table 4.1 (Continued)

	Wing Area ft^2	Tail Area ft^2	Wing M.A.C. ft
1.	231.77	n.a.	7.00
2.	303.30	n.a.	7.67
3.	452.0	n.a.	9.33
4.	542.5	n.a.	10.10
5.	980.0	257.0	12.00
6.	980.0	n.a.	11.28
7.	1,238.0	n.a.	19.90
8.	1,579.0	n.a.	15.38
9.	1,358.0	310.0	17.20
10.	1,700.0	376.0	18.18
11.	2,433.0	500.0	20.80
12.	2,521.0	545.0	20.92
13.	2,773.0	n.a.	22.20
14.	2,932.0	638.0	n.a.
15.	2,773.0	n.a.	22.20
16.	3,228.0	n.a.	22.20
17.	3,010.0	625.0	23.80
18.	5,500.0	1,470.0	38.30
19.	6,200.0	n.a.	33.20

Table 4.1 (Continued)

	Tail M.A.C. ft.	Wing Semi-Span	Tail Semi-Span
1.	3.50	17.79	7.33
2.	3.90	21.65	9.66
3.	3.63	26.25	10.66
4.	6.72	27.20	12.38
5.	9.35	44.25	14.75
6.	9.40	46.50	18.00
7.	8.90	47.57	15.08
8.	9.20	56.25	17.38
9.	9.38	44.92	17.12
10.	11.35	54.00	17.88
11.	14.00	65.42	19.83
12.	13.50	65.42	21.67
13.	10.90	71.20	23.75
14.	14.10	73.08	21.92
15.	10.90	71.20	23.75
16.	n.a.	79.96	25.19
17.	14.20	72.88	22.88
18.	23.50	97.83	36.37
19.	14.15	111.35	34.35

Table 4.1 (Continued)

	Tail to Wing Chord Ratio	Wing Taper Ratio	Rel. mass μ
1.	0.500	0.508	566
2.	0.510	0.333	510
3.	0.602	n.a.	443
4.	0.665	0.373	529
5.	0.780	0.323	470
6.	0.830	0.339	590
7.	0.446	0.221	275
8.	0.598	0.354	295
9.	0.546	0.268	347
10.	0.625	0.304	365
11.	0.672	0.333	325
12.	0.645	0.332	343
13.	0.491	0.230	314
14.	n.a.	n.a.	n.a.
15.	0.491	0.230	362
16.	n.a.	n.a.	310
17.	0.596	0.275	321
18.	0.614	0.246	228
19.	0.426	0.336	248

4.4 Parameters of the Second Type

The geometrical parameters have all been nondimensionalized through the use of the mean aerodynamic chord. The parameters are, therefore, defined as

$$\bar{b} = \bar{c}_t/\bar{c} \tag{4.3}$$

$$\hat{\lambda}_1 = \lambda_1/\bar{c} \tag{4.4}$$

$$\hat{\lambda}_2 = \lambda_2/\bar{c} \tag{4.5}$$

$$\hat{\lambda}_3 = \lambda_3/\bar{c} \tag{4.6}$$

$$\hat{\lambda}_4 = \lambda_4/\bar{c} \tag{4.7}$$

It should be noted that $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_3$, and $\hat{\lambda}_4$ represent distances from the craft's center of gravity to the location of the unsteady resultant forces acting on the wing and tail, as shown in Figure 2.1. They will be functions of spanwise position if the aircraft has a sweptback wing and tail and in such cases will be parameters of the third type.

The characteristics of parameters of the second type are shown in Figure 4.6. They have a constant value over the entire range of aircraft considered.

The reduced frequency parameter for the wing is defined as $k = \frac{\omega\bar{c}}{2u_o}$ while that for the tail is similarly defined as $k_t = \frac{\omega\bar{c}_t}{2u_o}$. The ratio of the frequency parameters is, therefore, equal to the ratio of the mean aerodynamic chords shown in Figure 4.6 to be 0.6.

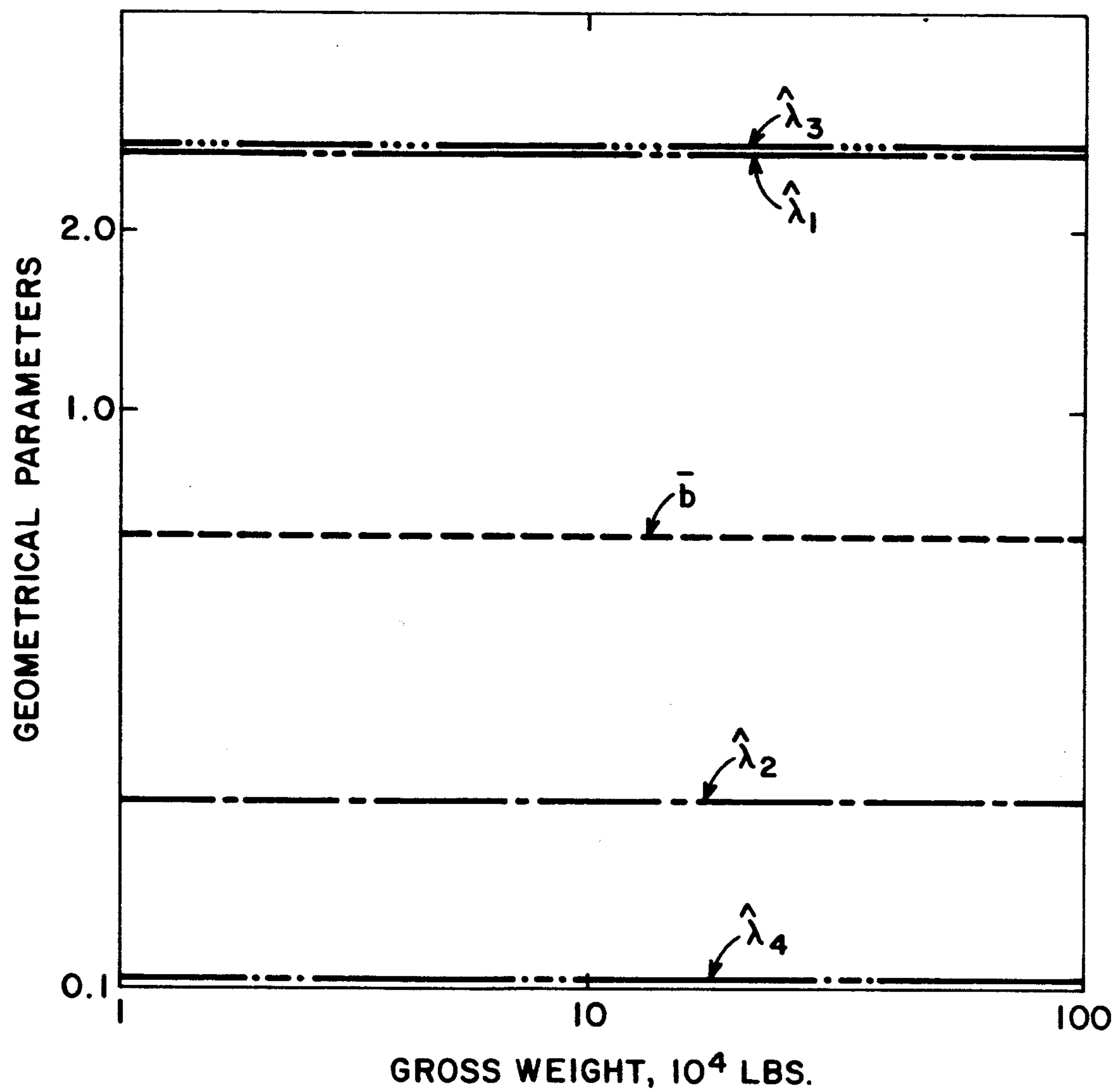

Figure 4.6 Computed Values for Geometrical Aircraft Parameters

4.5 Parameters of the Third Type

Although assumed not to apply in the numerical analysis of this study, parameters of the third type occur due to change in a geometrical parameter with spanwise location.

The first case is that of a tapered wing and tail. The airfoil chord at any spanwise location may be expressed simply as the chord at the root multiplied by a coefficient a(y). The relation is therefore,

$$C_p(y) = a(y)\, C_r \tag{4.8}$$

The wing taper coefficient a(y) is best expressed in terms of another coefficient C_1 in the form,

$$a(y) = 1 - C_1 y \tag{4.9}$$

where C_1 is expressible as,

$$C_1 = \frac{(\lambda - 1)}{\ell} \tag{4.10}$$

Similarly for the tail, the following may be written.

$$C_{P_t}(y) = a_t(y)\, C_{r_t} \tag{4.11}$$

$$a_t(y) = 1 - C_2 y \tag{4.12}$$

$$C_2 = \frac{(\lambda_t - 1)}{\ell_t} \tag{4.13}$$

The trend of the coefficients C_1 and C_2 are shown in Figure 4.7. Another contributing case to this third type of parameter is the swept wing case. In this case any distances from a fixed location on the

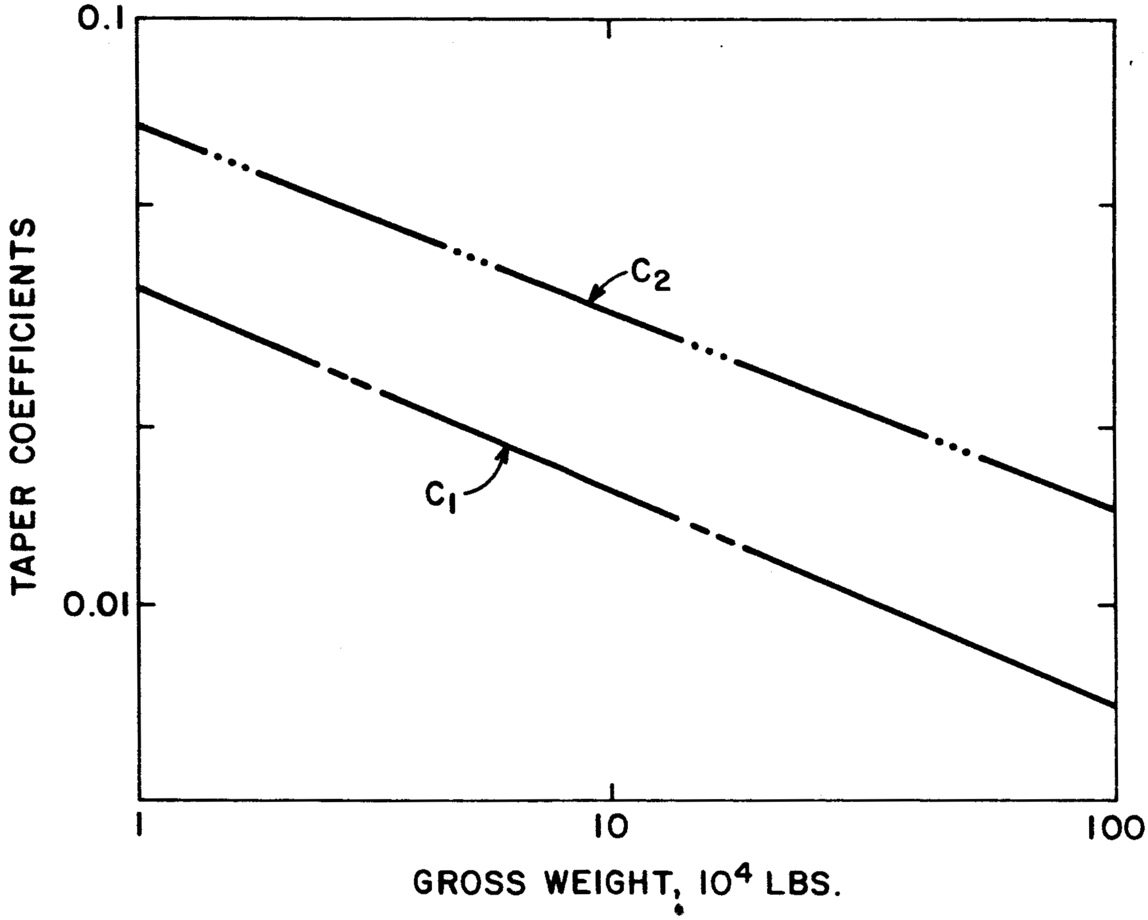

Figure 4.7 Wing and Tail Taper Coefficients

airfoil now become functions of y. As an example, the distances $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_3$, and $\hat{\lambda}_4$, which for a straight wing craft are invariant, now become

$$(4.14) \qquad \hat{\lambda}_1(y) = \hat{\lambda}_{o_t} + \frac{y \tan \Lambda \, 1/4}{\bar{c}}$$

$$(4.15) \qquad \hat{\lambda}_2(y) = \hat{\lambda}_{o_w} - \frac{y \tan \Lambda \, 1/4}{\bar{c}}$$

$$(4.16) \qquad \hat{\lambda}_3(y) = \hat{\lambda}'_{o_w} + \frac{y \tan \Lambda \, 1/4}{\bar{c}}$$

$$(4.17) \qquad \hat{\lambda}_4(y) = \hat{\lambda}'_{o_w} - \frac{y \tan \Lambda \, 1/4}{\bar{c}}$$

The fundamental mode shape for elastic bending can be approximated by the expression

$$(4.18) \qquad \phi_2(y) = (1 - \cos \frac{\pi y}{2\ell})$$

and the fundamental torsional mode shape by

$$(4.19) \qquad \psi_2(y) = \sin \frac{\pi y}{2\ell}$$

4.6 Exhibited Trends

It is an extremely convenient and useful result that the various dimensionless parameters have such a simple form for conventional aircraft. This greatly generalizes the resulting equations of motion.

In summary, the following characteristics are observed: Those parameters which depend on mass and mass distribution show a linear decreasing trend with increasing gross weight. Parameters which are geometrical without spanwise dependency are invariant over the entire weight range of craft. Geometrical parameters which are functions of

spanwise distance may be defined in terms of coefficients which are found to exhibit a linear decreasing trend with increasing gross weight. Parameters which are geometrical without spanwise dependence are invariant over the entire weight range of craft.

4.7 Bending to Torsion Frequency Ratio

The equations of motion for elastic bending and torsion include the ratios of bending natural frequency to the forcing frequency (ω_h/ω), and torsional natural frequency to the forcing frequency (ω_α/ω). First, the value of the ratio of bending to torsion frequency (ω_h/ω_α) will be established.

This value may be obtained approximately by considering a cantilever wing. The expression for the elastic bending natural frequency is

$$\omega_h = \frac{c_h}{\ell 2}\sqrt{\frac{EI}{m}} \tag{4.20}$$

where c_h is a coefficient whose value depends upon the mode considered. For the fundamental mode, its value is 3.5.

The expression for the torsional natural frequency is

$$\omega_\alpha = \frac{c_\alpha}{\ell}\sqrt{\frac{GJ}{I_o}} \tag{4.21}$$

where again c_α is dependent upon the torsional mode considered. The value of c_α for the fundamental mode is $\pi/2$.

The ratio of frequencies is, therefore, given by

$$\frac{\omega_h}{\omega_\alpha} = 2.22\sqrt{\frac{EII_o}{GJm\ell^2}} \tag{4.22}$$

The shear modulus G may be related to the Young's modulus E through the relation,

(4.23) $E = 2(1 + \nu)G$

where ν is the Poisson ratio for the material, and is found to be about 0.25 for aircraft materials. Therefore,

(4.24) $E = 2.5G$

The bending moment of inertia I is a geometrical quantity which may be expressed as:

(4.25) $I = c_1 \bar{c} t^3$

The other geometrical term in the frequency relation is the polar moment of inertia J which derives its primary contribution from the moment of inertia about an axis perpendicular to the chord line. Therefore, J is expressible as,

(4.26) $J = c_2 t \bar{c}^3$

The coefficient c_1 is approximately equal to coefficient c_2 for wing type sections. Since $c_1 = c_2$, the ratio of the section moment of inertia may be expressed as,

(4.27) $(\frac{K}{J}) \simeq (\frac{t}{\bar{c}})^2$

The wings of subsonic jets have a thickness to chord ratio of 0.13, with few exceptions.

The mass moment of inertia of a wing may be written as,

(4.28) $I_o = c_3 m \bar{c}^2$

For a wing section the value of c_3 generally has a value of the order of 0.1. The ratio of $\bar{c}$ to ℓ for subsonic jets has a value of 0.40. Therefore the following may be written:

$$(4.29) \qquad \left(\frac{I_o}{m\ell^2}\right) \simeq 0.1 \left(\frac{\bar{c}}{\ell}\right)^2$$

Substituting equations (4.24) through (4.29) into equation (4.22) yields:

$$(4.30) \qquad \frac{\omega_h}{\omega_\alpha} \simeq 1.13\ (t/\ell)$$

A reasonable approximation to the frequency ratio is, therefore:

$$(4.31) \qquad \frac{\omega_h}{\omega_\alpha} \simeq 0.06$$

Credibility is added to this value by an analysis performed by Houbolt and Anderson [18] 1948, who calculated the frequencies of the fundamental modes of a nonuniform beam. They found, using the Stodola method of frequency determination, that the bending frequency had the value in the fundamental mode of 2.4 rad per sec. The fundamental torsional mode frequency is given as approximately 71.0 rad per second. This would indicate a value for the frequency ratio of 0.035, which is the same order of magnitude as is determined by this analysis.

The effect of a variation in the frequency ratio (ω_h/ω_α) was established numerically. Computations were performed with a range of (ω_h/ω_α) from 0.01 to 1.00 in the equations of motion. It was found that no shift in any resonant peak resulted. Slight variation in the magnitude did result, however, no consistent trend was verified and in all cases, the variation was less than ten percent of the magnitudes resulting from $(\omega_h/\omega_\alpha) = 0.06$.

4.8 Relation of Natural Frequency to Aircraft Physical Parameters

It would be advantageous to express the bending and torsional natural frequencies in terms of basic physical quantities and known aircraft parameters. Beginning with the relation for the bending natural frequency of a cantilever wing, the first mode is expressed,

$$\omega_h = \frac{3.5}{\ell^2}\sqrt{\frac{EI}{m}} \tag{4.32}$$

After appropriate substitutions, the following relation is obtained:

$$\omega_h = 1.758\sqrt{\mu\left(\frac{E}{\sigma_b}\right)\left(\frac{\rho}{\rho_s}\right)\left(\frac{\bar{c}}{\ell}\right)\left(\frac{g}{\ell}\right)\left(\frac{c_1'}{c_2'}\right)} \tag{4.33}$$

The coefficients c_1' and c_2' derive their values from the evaluation of the following expressions. The bending moment of inertia is expressed as,

$$I = \frac{c_1' \, W \, \ell t}{\sigma_b} \tag{4.34}$$

and the mass distribution is expressed as,

$$m = c_2' \, \rho_s \, t\bar{c} \tag{4.35}$$

Evaluation of coefficients in terms of those aircraft values which will be known for a given craft yields the expression for the coefficients as,

$$c_1'/c_2' = \frac{\sigma_b \, I \, \rho_s \bar{c}}{W\ell m} \tag{4.36}$$

which may be written in terms of a coefficient previously introduced as,

$$(4.37) \qquad \frac{c_1'}{c_2'} = \frac{0.39\, c_1 \sigma_b\, \rho_s\, \bar{c}t^3}{Wm}$$

The expression for frequency may also be stated in terms of aircraft parameters, values of which are given by Werner [19]. The expression for frequency is;

$$(4.38) \qquad \omega_h = \frac{c'''}{\ell^2}\, c_\sigma \left(\frac{c_w c_m}{c_b}\right)^{3/2} \sqrt{\left(\frac{E}{\sigma_b}\right) \left(\frac{W}{W_s}\right)^3 \left(\frac{A}{\bar{c}t}\right)^3 \left(\frac{\ell}{\bar{c}}\right) \left(\bar{c}^3 g\right)}$$

Figures 4.8 through 4.10 give the Werner subsonic jet values.

$$(4.39) \qquad c_\sigma \left(\frac{c_w c_m}{c_b}\right)^{3/2} = 0.8\ (10^{-4})$$

$$(4.40) \qquad \left(\frac{W}{W_s}\right) = \text{(as shown in Figure 4.2)}$$

$$(4.41) \qquad \left(\frac{A}{\bar{c}t}\right) = \frac{1}{2.4}\ (10^{-2})$$

The value of $(\bar{c}/\ell)$ may be taken as 0.40 for subsonic jet aircraft.

The frequency for torsion may also be expressed in terms of several nondimensional parameters. Using the relation,

$$(4.42) \qquad \omega_\alpha = \frac{\pi}{2\ell} \sqrt{\frac{GJ}{I_o}}$$

substitution of the following yields the required relation.

$$(4.43) \qquad J = \frac{c_1''\, W\, \bar{c}^2}{\sigma_s}$$

$$(4.44) \qquad I_o = c_2''\, \rho_s \bar{c}^3 t$$

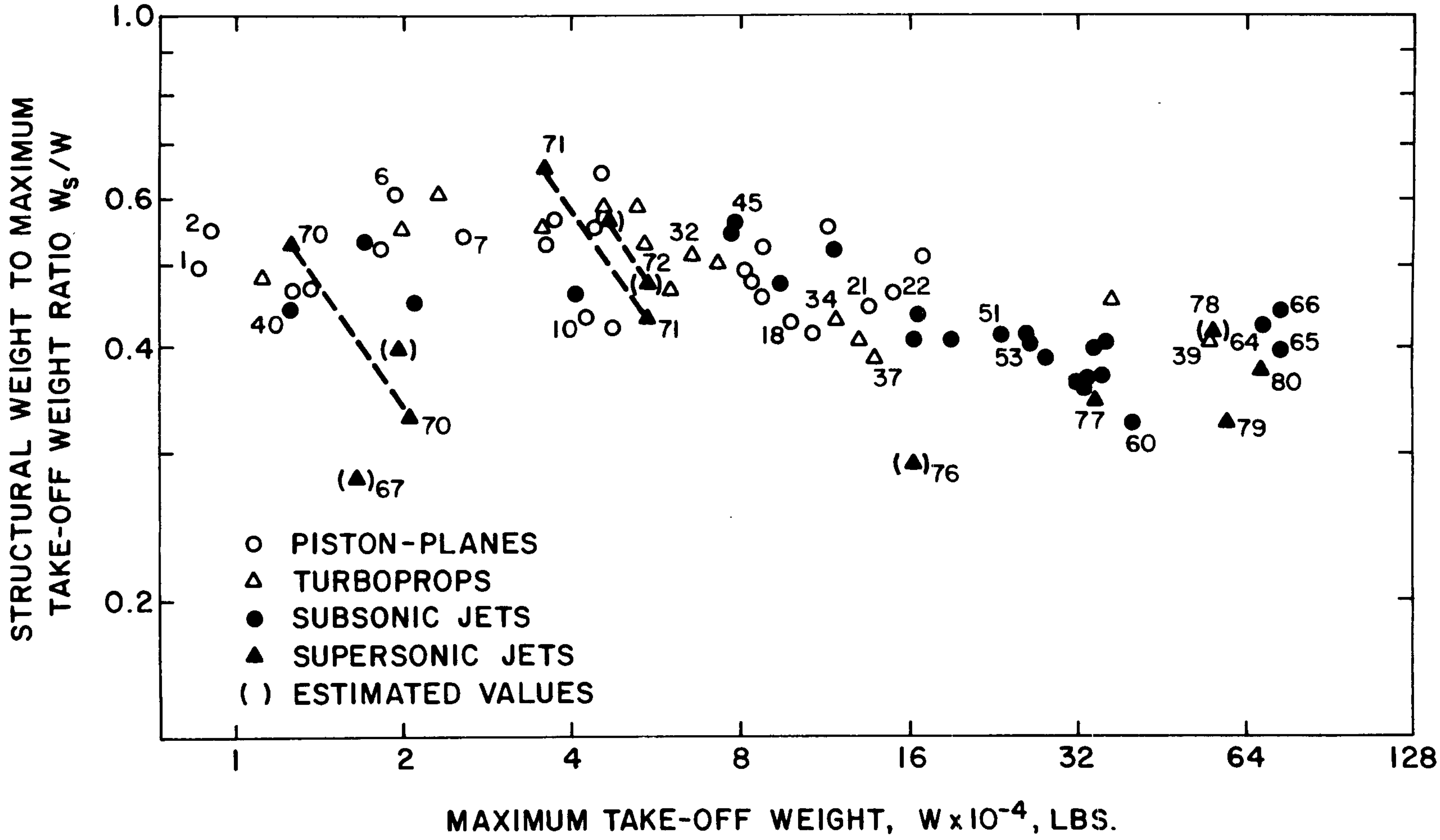

Figure 4.8 Structural Weight to Maximum Take-off Weight Ratio

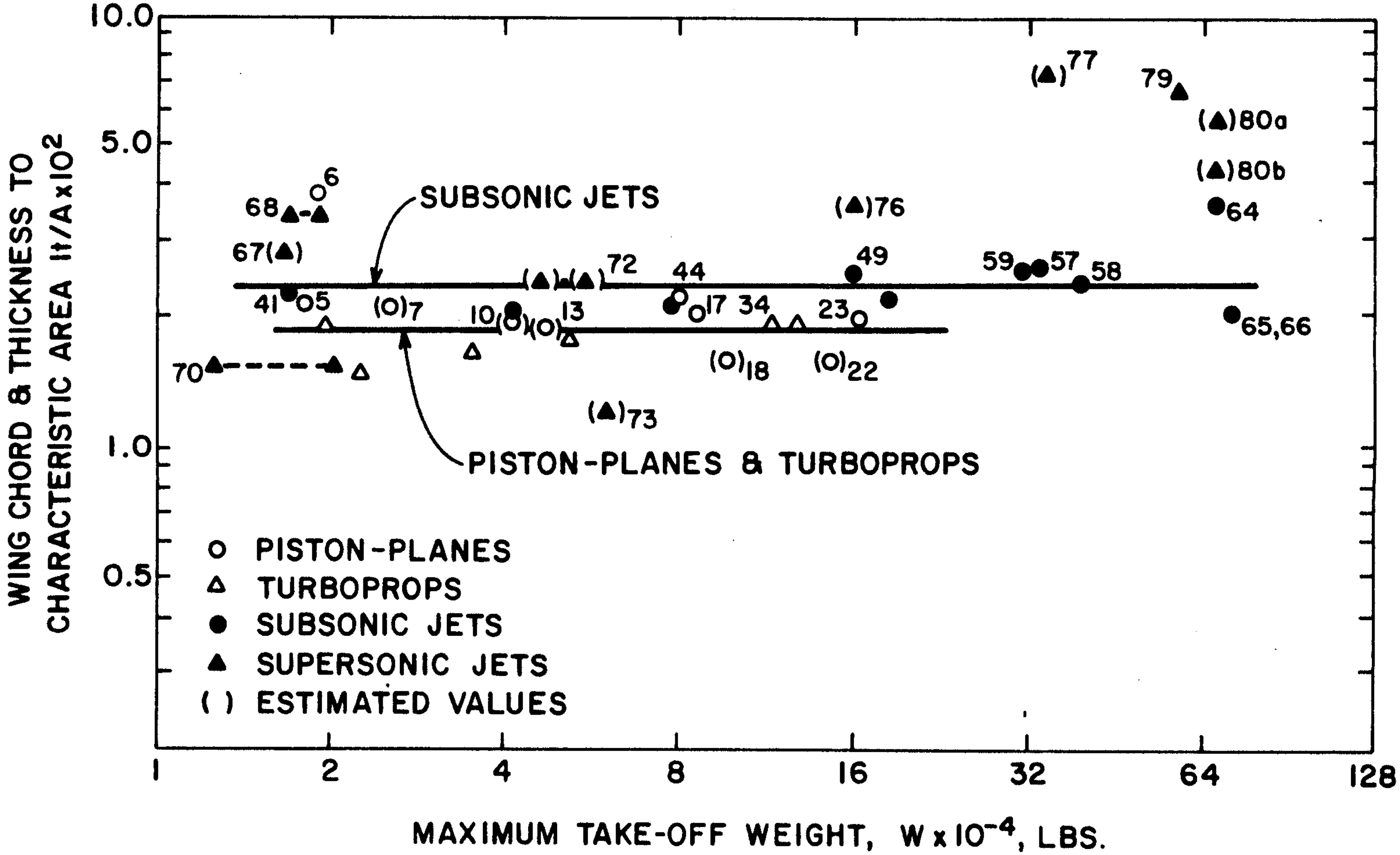

Figure 4.9 Wing Chord and Thickness to Characteristic Area

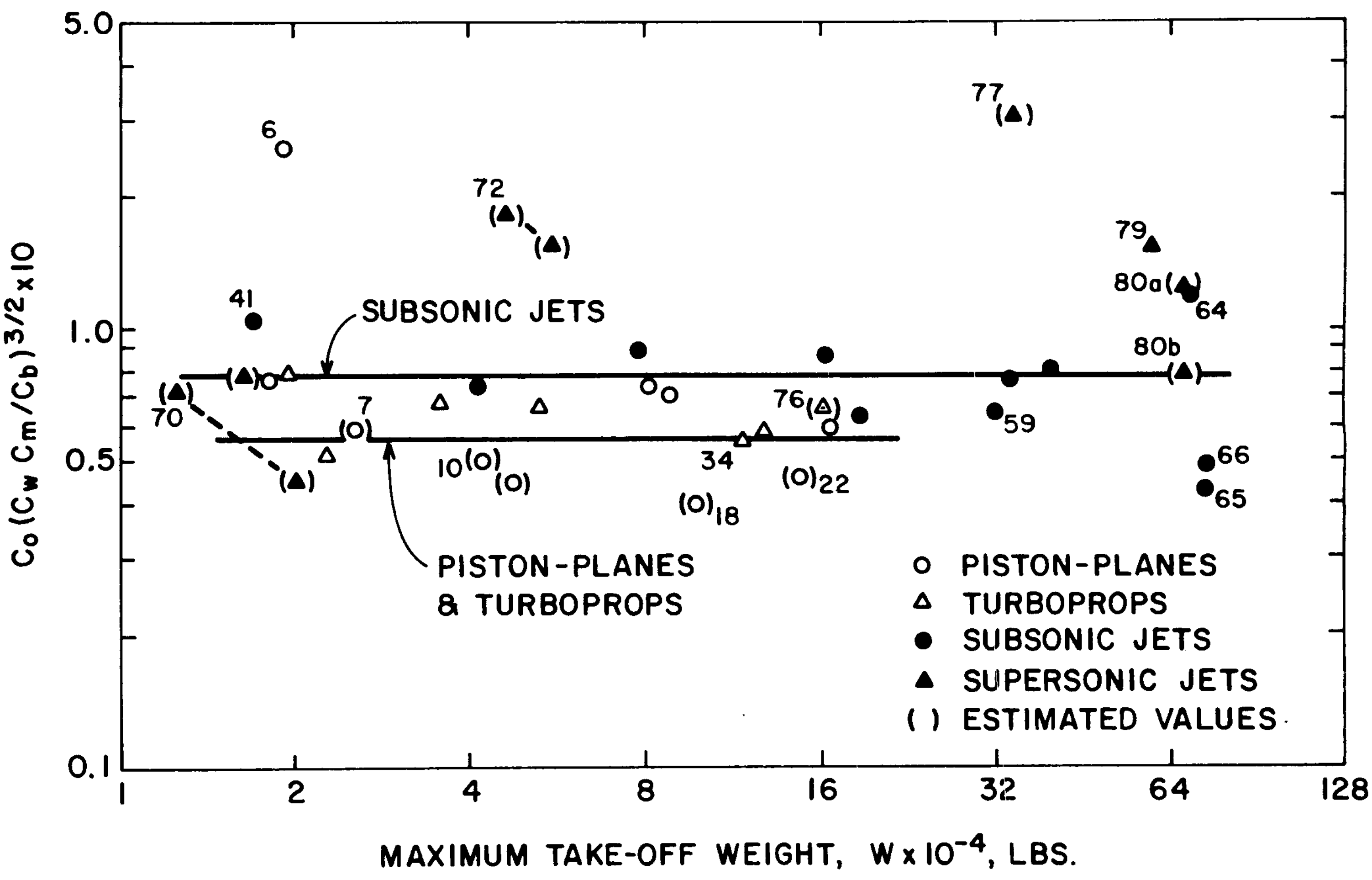

Figure 4.10 $C_\sigma \ (C_w C_m / C_b)^{3/2}$

The relation is,

$$(4.45) \qquad \omega_\alpha = 17.75 \sqrt{\mu(\frac{G}{\sigma_s}) (\frac{\rho}{\rho_s}) (\frac{\bar{c}}{\ell}) (\frac{g}{\ell}) (\frac{c_1''}{c_2''})}$$

Again the constants may be evaluated from equations (4.43) and (4.44). It may be seen therefore, that once the nondimensional ratios of equations (4.33) and (4.45) are known, it is merely a matter of dictating a relative mass in order that the natural frequencies become known.

The evaluation of the ratio (c_1''/c_2'') yields the relation

$$(4.46) \qquad c_1''/c_2'' = \frac{\sigma_s \rho_s \bar{c} t J}{W I_o}$$

In terms of the parameters as given by Werner, the relation for torsional frequency is,

$$(4.47) \qquad \omega_\alpha = c_3'' c \left(\frac{c_w c_m}{c_b}\right)^{3/2} \sqrt{(\frac{W}{W_s})^3 (\frac{G}{\sigma_s}) (\frac{A}{\ell t})^3 (\frac{\ell}{t}) (\frac{g}{\bar{c}})}$$

Using relations (4.33) and (4.45), the ratio of the frequencies may now be stated in the form,

$$(4.48) \qquad \frac{\omega_h}{\omega_\alpha} = c_4 \sqrt{\frac{t}{\ell} \frac{A^{1/2}}{\ell}}$$

where A is the characteristic area of the aircraft.

For subsonic jet aircraft, both (t/ℓ) and $(A^{1/2}/\ell)$ are nearly constant over the entire range of craft. Therefore, it may be concluded that the ratio ω_h/ω_α is a constant over the entire range. The value previously presented, ω_h/ω_α, equal to 0.06 is valid for all craft within the group.

5.0 NUMERICAL RESULTS AND CONCLUSIONS

5.1 Power Spectral Techniques

The equations formulated in Chapters 2.0 and 3.0 will yield the frequency response functions of the modes considered. For a stationary random process, power spectral techniques may be used to determine the power spectral density of output quantities once input spectral densities are established. Given a random input or output function, the power spectral density is defined as the Fourier transform of the auto-correlation function [25]. The power spectral density of the input may be related to the power density of the output by the well known expression:

$$(5.1) \qquad \Psi'(k) = |H(ik)|^2 \, \Phi'(k)$$

where $\Psi'(k)$ and $\Phi'(k)$ are the respective spectral densities. The response of the system depends on the input spectral density and secondly on the characteristics of the mechanical admittance squared.

Mathematical expressions frequently used to represent power spectral density curves contain three parameters. The root mean square velocity measures the intensity of the turbulence. When the power spectral density is plotted against frequency, the r.m.s. has the effect of moving the curve upward or downward parallel to the ordinate. A second parameter determines the slope at the high frequency end of the curve. The third parameter, scale length, is a shape parameter defining the rate change of the slope in the low frequency region of the curve.

Relations for the input spectral density of atmospheric turbulence have been presented by numerous investigators. Von Karman [4] 1940, presented an expression for the power spectral density of atmospheric turbulence based on a high frequency slope of-(5/3). Crooks, Hoblit, and Prophet [32] 1967, have presented power spectral density relations appropriate to the case of high altitude clear air turbulence. Also, a study by Ashburn, Waco, and Mitchell [33] 1969, presents models of high altitude clear air turbulence. The scale length parameter distribution at high altitudes has been studied by Ashburn [34] 1969. Most recently an expression for the input power spectral density has been presented by Houbolt [15], who presents aircraft structural design procedures based on this relation. The relation and its plot are shown in Figure 5.1.

The purpose of this analysis is to present the characteristics of the mechanical admittance functions which generally depend on the relative mass parameters, Mach number, and the frequency at which the aircraft is being forced. All other aircraft parameters may either be related to one of these or are constant, as has been shown in Chapter 4.0.

5.2 Mechanical Admittance Results

For the class of subsonic transport jets considered in this analysis, the relative mass parameter has a range of two hundred (200) to six hundred (600). Equations (2.29), (2.30), (3.13), and (3.14) may be solved simultaneously to obtain the mechanical admittance values. If equations (3.13) and (3.14) are divided by (k^2), the ratio of bending to forcing reduced frequency (k_h/k), and torsion to forcing reduced frequency (k_α/k), occur. In order to solve the equations, appropriate

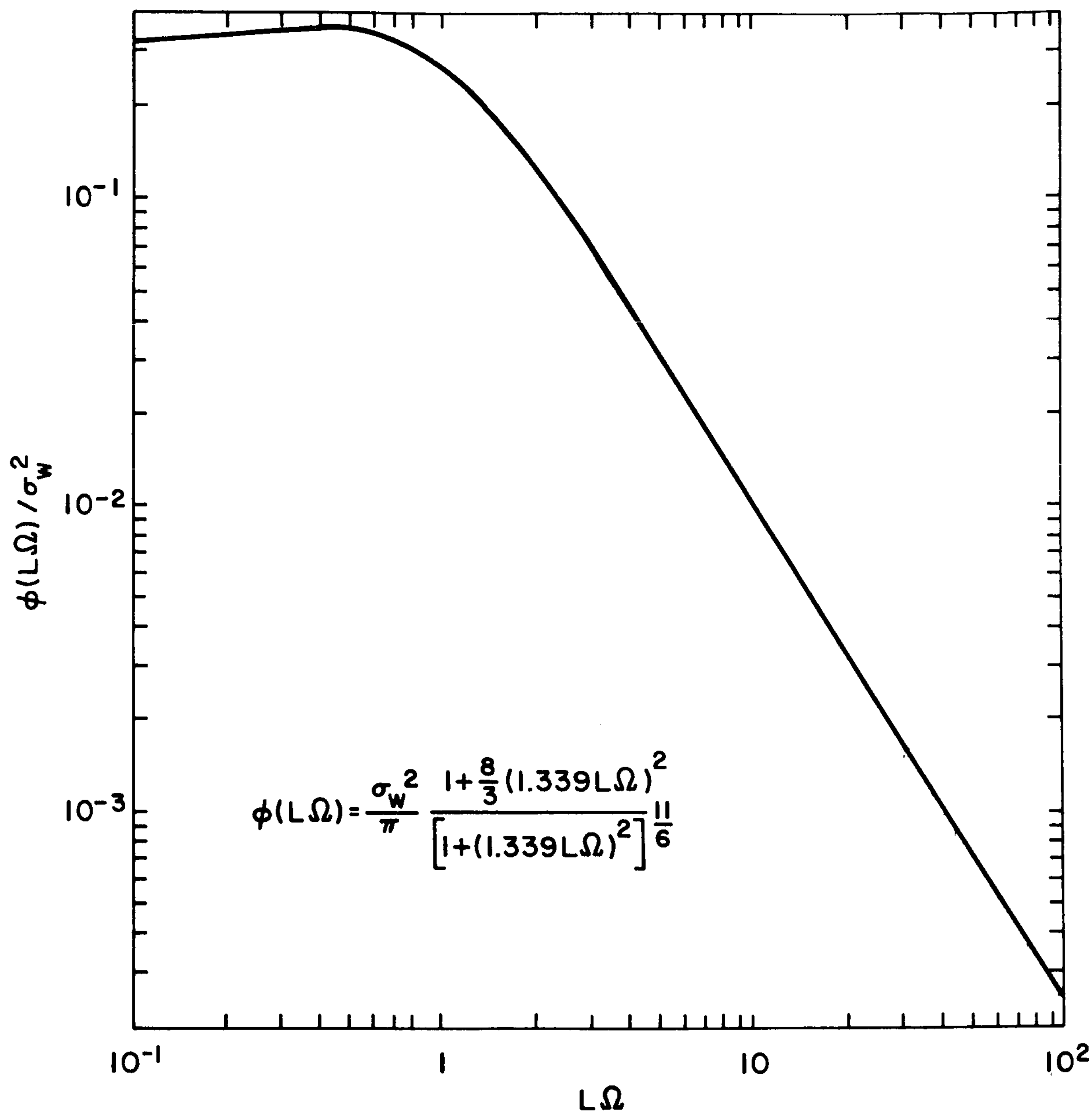

Figure 5.1 Gust Spectrum Function

values must be obtained for these ratios in terms of those quantities defined in Chapter 4.0.

The expression for torsional frequency was given previously as:

$$(4.20) \qquad \omega_\alpha = \frac{c_\alpha}{\ell} \sqrt{\frac{GJ}{I_o}}$$

The ratio of torsional frequency to forcing frequency may be expressed as follows:

$$(5.2) \qquad \frac{k_\alpha}{k} = \frac{c'_\alpha}{k} \sqrt{\frac{GJ}{I_o}}$$

where the coefficient c'_α is given as

$$(5.3) \qquad c'_\alpha = 0.3 \left(\frac{1}{a'M}\right)$$

The quantity a' is the speed of sound at the flight altitude and M is the Mach number.

In Chapter 4.0, the expression relating the polar moment of inertia to the area moment of inertia about the chordwise axis was expressed as

$$(4.27) \qquad \left(\frac{J}{I}\right) = \left(\frac{\bar{c}}{t}\right)^2$$

Using equations (4.25) and (4.28), the ratio of the area moment of inertia to the mass moment of inertia becomes,

$$(5.4) \qquad \left(\frac{I}{I_o}\right) = \frac{t^3}{m\bar{c}}$$

The quantity (J/I_o), therefore, is written;

$$(5.5) \qquad (J/J_o) \simeq \left(\frac{\bar{c}t}{m}\right)$$

The nondimensional wing mass distribution may be incorporated into equation (5.5) to yield;

$$(5.6) \qquad \left(\frac{J}{I_o}\right) = \frac{t}{\rho \bar{m} \ell}$$

Substitution of equation (5.6) into equation (5.2) yields for the required ratio,

$$(5.7) \qquad \frac{k_\alpha}{k} = \frac{c''_\alpha}{k} \sqrt{\frac{G}{\bar{\bar{m}}}}$$

where c''_α is a function of altitude and Mach number, and may be written as

$$(5.8) \qquad c''_\alpha = 0.067 \left(\frac{1}{a'M}\right) \sqrt{\frac{1}{\rho}}$$

Values for $\bar{m}$, the nondimensional wing mass distribution were given in Chapter 4.0.

The ratio of the bending reduced frequency parameter to the forcing reduced frequency may be stated:

$$(5.9) \qquad \left(\frac{k_h}{k}\right) = \left(\frac{k_h}{k_\alpha}\right) \left(\frac{k_\alpha}{k}\right)$$

where (k_h/k_α) was determined to be approximately 0.06 in Chapter 4.0.

The results of numerical computations are tabulated in Appendix B for the cases of Mach number equal to 0.2, 0.5, and 0.7. Figures 5.2 through 5.5 show the behavior of the mechanical admittances at a Mach number of 0.7.

Examination of Figures 5.2 through 5.5 show that secondary excitation occurs from modes other than the one being considered. In certain

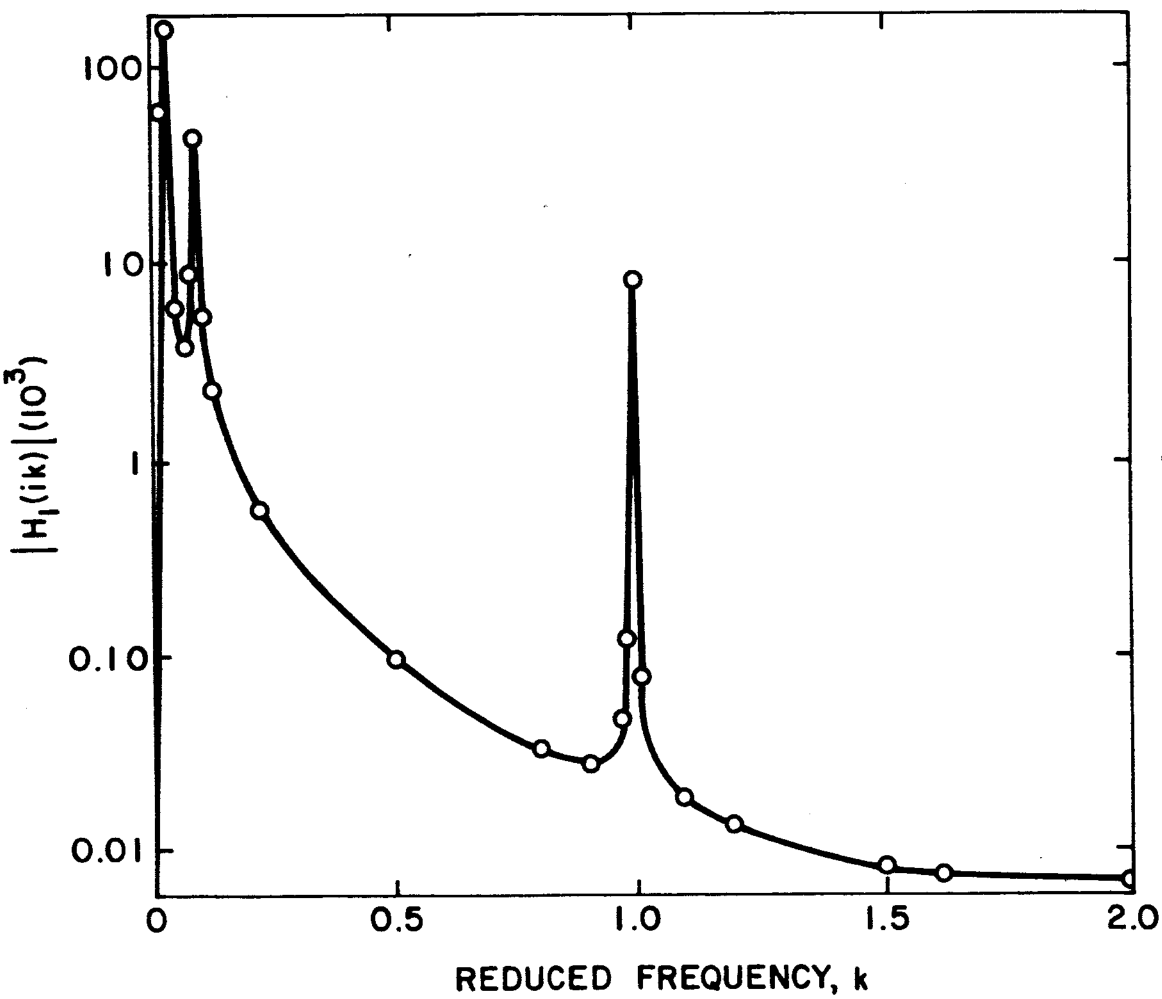

Figure 5.2 Rigid Body Plunging Mechanical Admittance
$\mu = 200$, $M = 0.7$

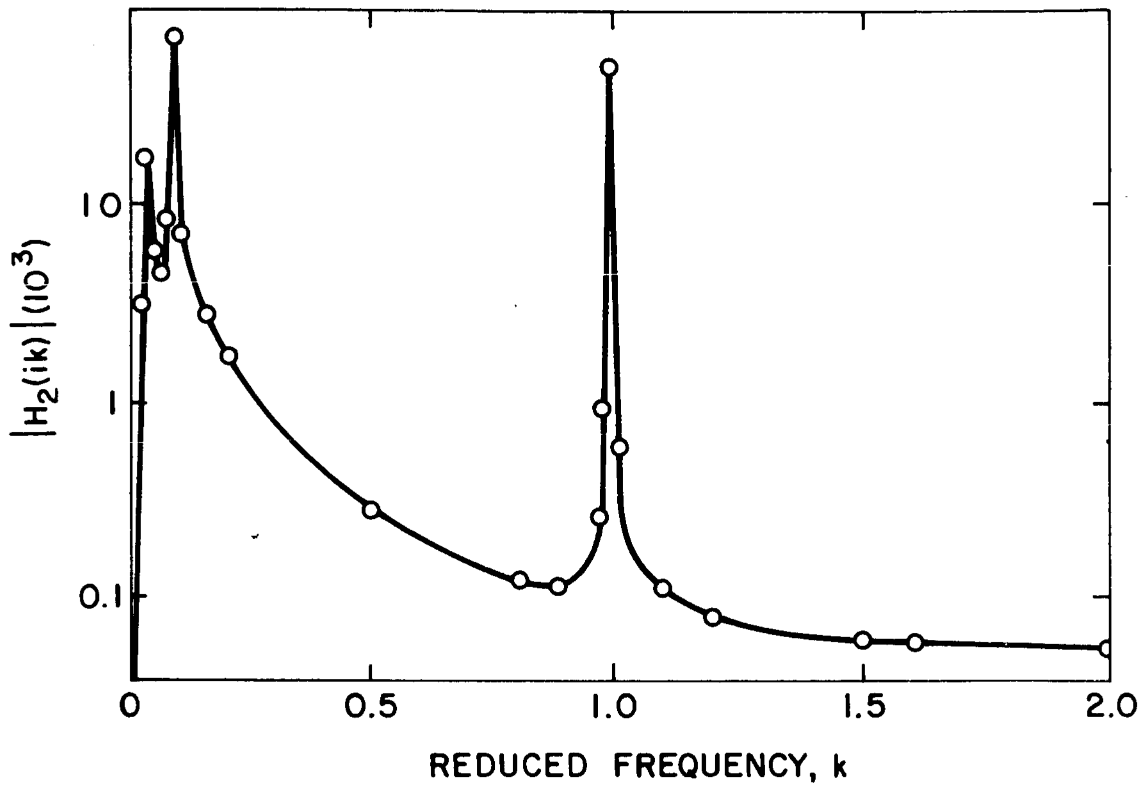

Figure 5.3 Wing Elastic Bending Mechanical Admittance
μ = 200, M = 0.7

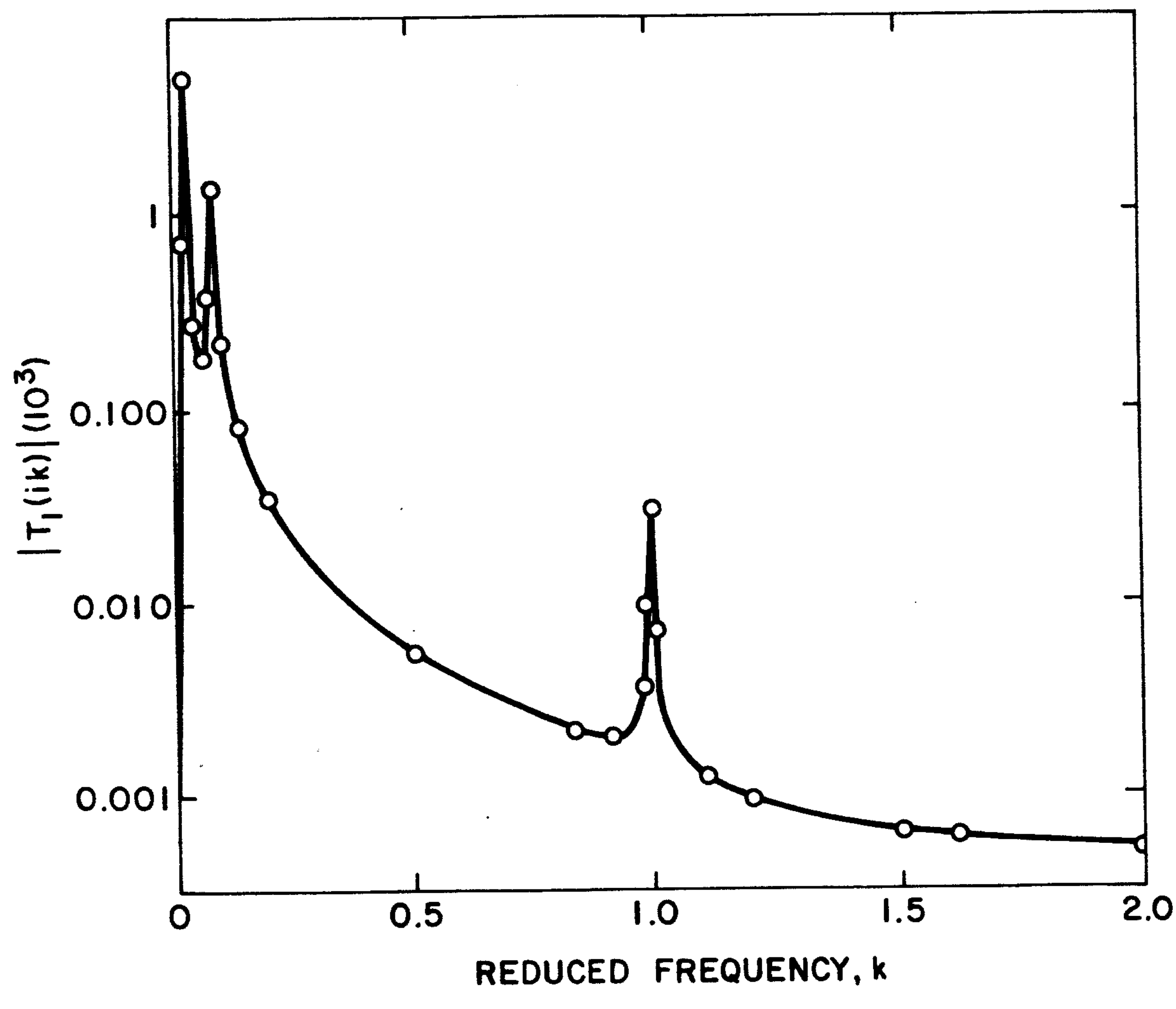

Figure 5.4 Rigid Body Pitching Mechanical Admittance
μ = 200, M = 0.7

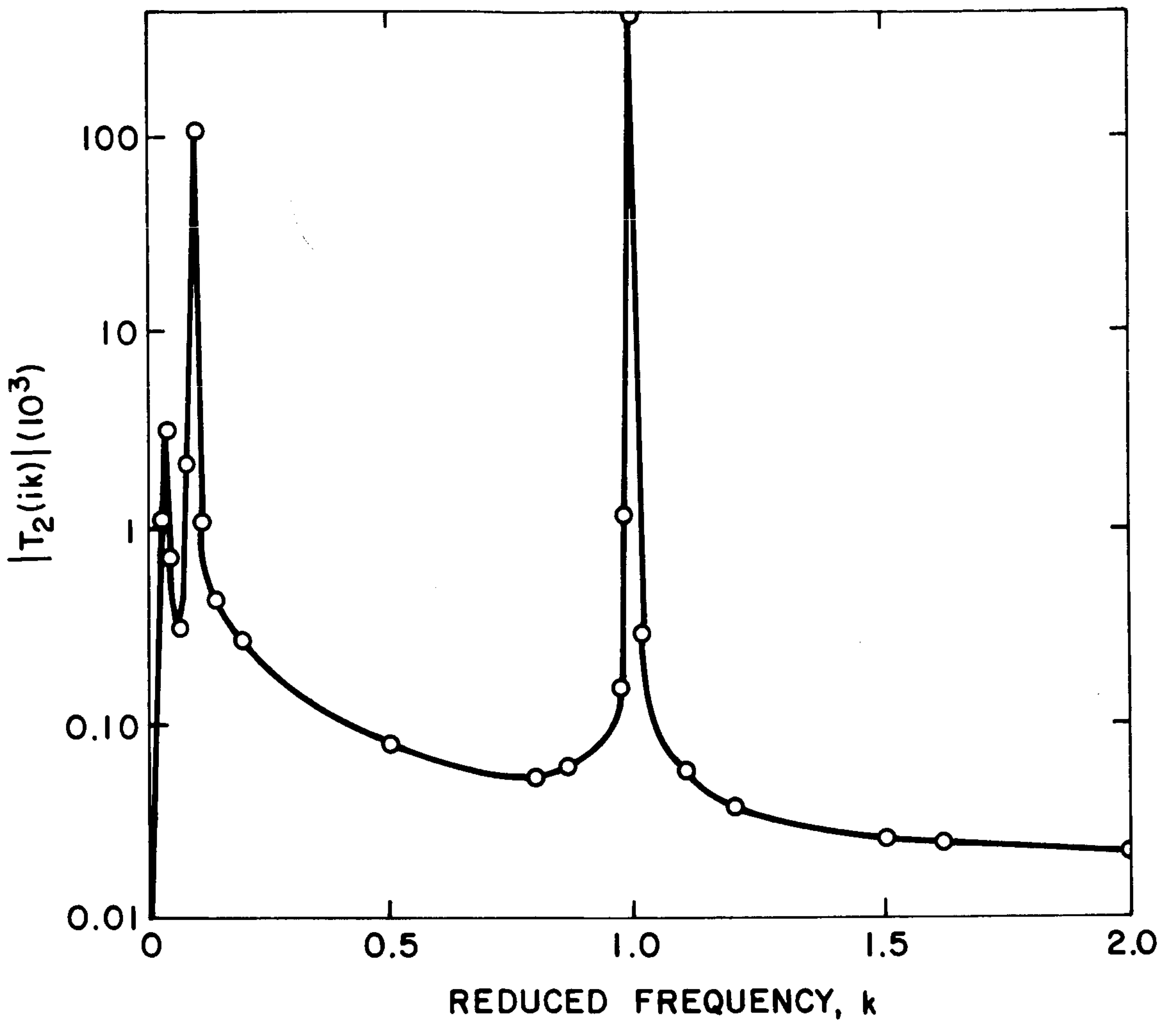

Figure 5.5 Wing Torsion Mechanical Admittance
$\mu = 200$, $M = 0.7$

instances, this secondary excitation is of nearly comparable amplitude to that at the excitation frequency for that mode. This would indicate that the detrimental frequencies in a particular mode will be the characteristic frequency to which that mode responds plus possibly all frequencies characteristic to the other modes of the problem.

The precision of the aerodynamic coefficients and other frequency dependent values could only be specified at intervals of reduced frequency equal to or greater than 0.01. Since both the plunging and pitching occur at relatively low frequencies, a highly detailed plot of each is impossible. The first resonant peak shown on Figures 5.2 through 5.5 includes the contribution to the respective admittance quantities from both of these modes. If smaller intervals had been achievable, then the pitching and plunging modes could have been extracted separately. As shown in this analysis, they appear together and no differentiation have been made between them. The critical values for both have been shown to be the same.

5.3 Variation of Resonant Reduced Frequency with Mach Number and Relative Mass

It is found that the resonant reduced frequencies remain nearly constant over the range of relative mass as shown in Figures 5.6 through 5.8. These values are taken from the numerical results. A minor shift of the resonant reduced frequency to a slightly lower value does occur with increasing relative mass; however, the order of magnitude of the shift is less than $\pm$ 0.005. The effect of this shift is neglected and the frequency will be treated as invariant.

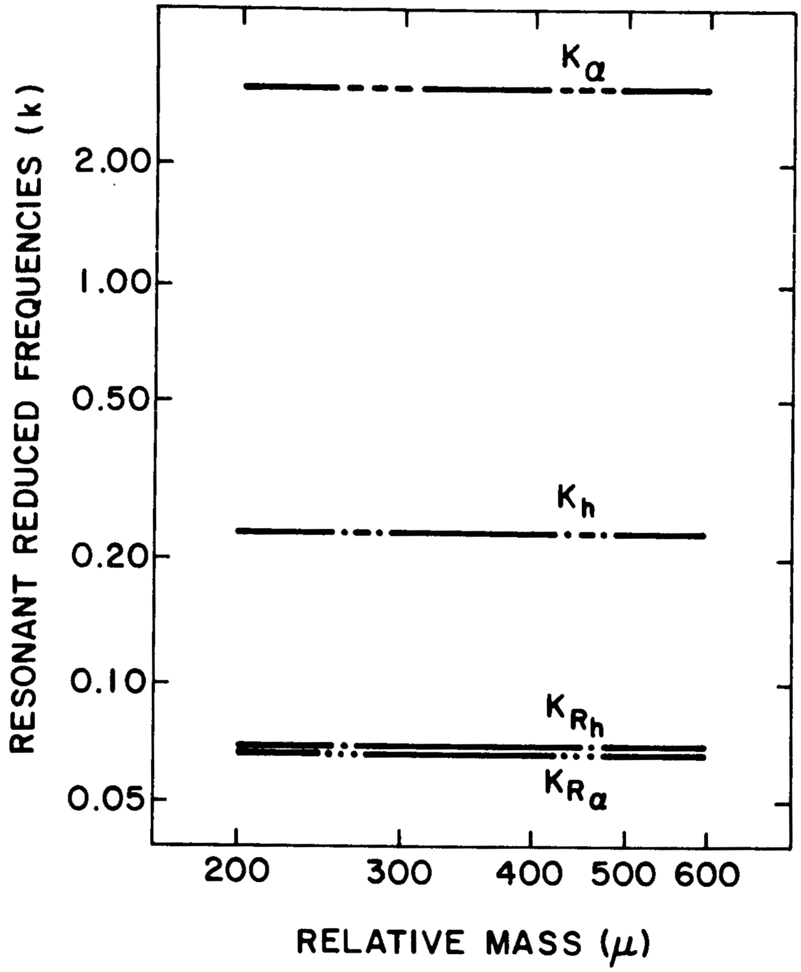

Figure 5.6 Resonant Reduced Frequency Variation with Relative Mass M = 0.2

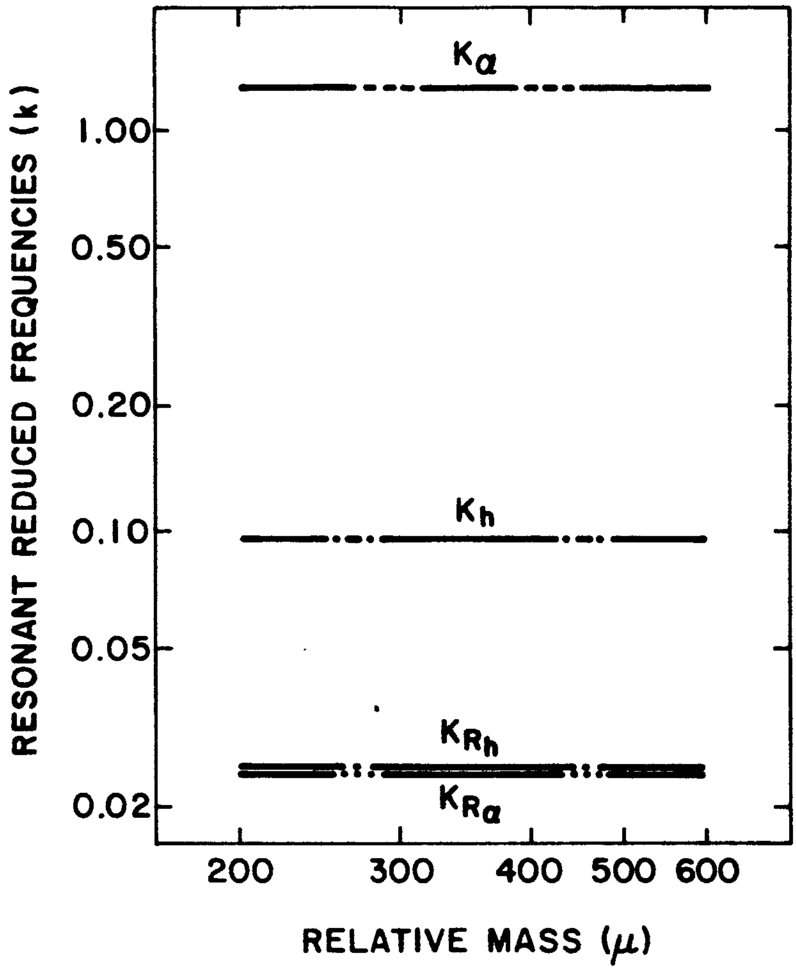

Figure 5.7 Resonant Reduced Frequency Variation with Relative Mass M = 0.5

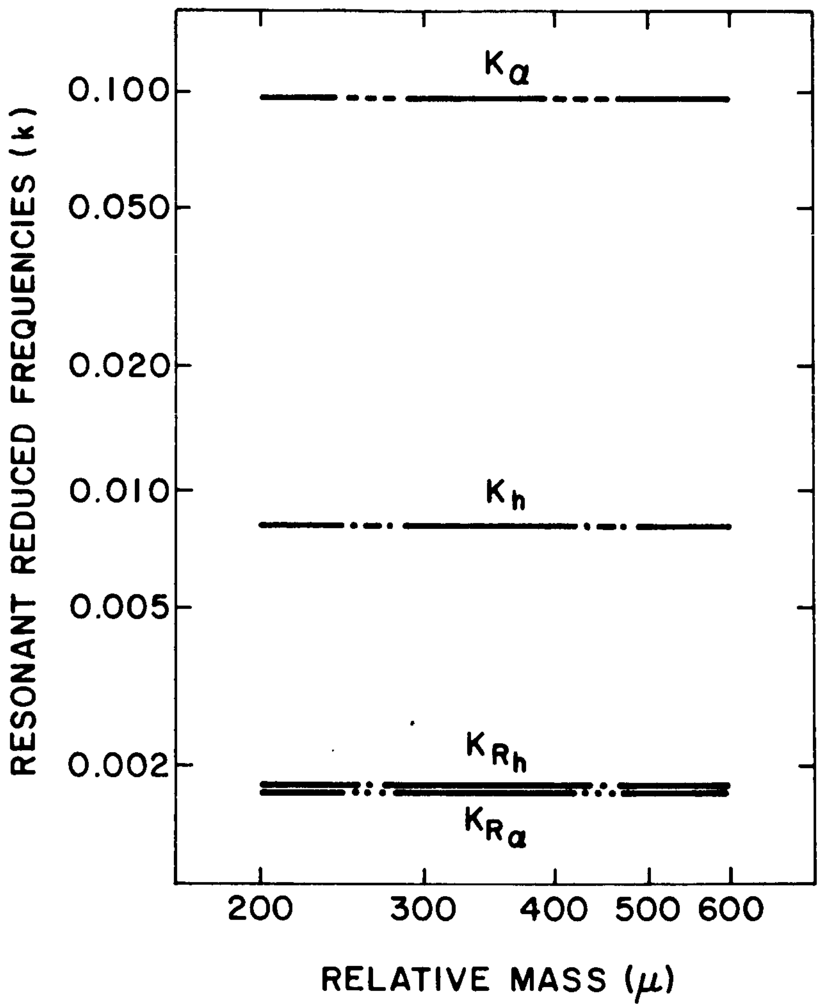

Figure 5.8 Resonant Reduced Frequency Variation with Relative Mass M = 0.7

The resonant reduced frequency is, however, dependent on the Mach number. This dependence shows a rapid growth of k as Mach number approaches zero. This would be expected from equation (5.3), which shows the frequency coefficient to be inversely proportional to Mach number. The variation of Mach number is illustrated in Figures 5.9 through 5.12. For Mach numbers of 0.2, 0.5, and 0.7, the resonant reduced frequencies have been established as shown in Table 5.1. The resonant frequencies for all Mach numbers are shown in Figures 5.13 through 5.16 as derived from the numerical computation.

Table 5.1 Resonant Reduced Frequencies

Mach Number	Plunging	Pitching	Bending	Torsion
0.2	0.065	0.065	0.250	3.290
0.5	0.025	0.025	0.098	1.290
0.7	0.018	0.018	0.076	1.000

5.4 Variation of Admittance Magnitudes with Mach Number and Relative Mass

Having considered the nature of the resonant frequencies and their variation with Mach number and relative mass, the final question is how the magnitudes of the response are affected by variations in Mach number and relative mass.

Figures 5.2 through 5.5 give the mechanical admittance values for the specific case of $M = 0.7$ and $\mu = 200$. It is found that the admittance functions for the plunging, pitching, and elastic bending modes decrease at $\mu = 600$ to approximately one third the value of the response at $\mu = 200$.

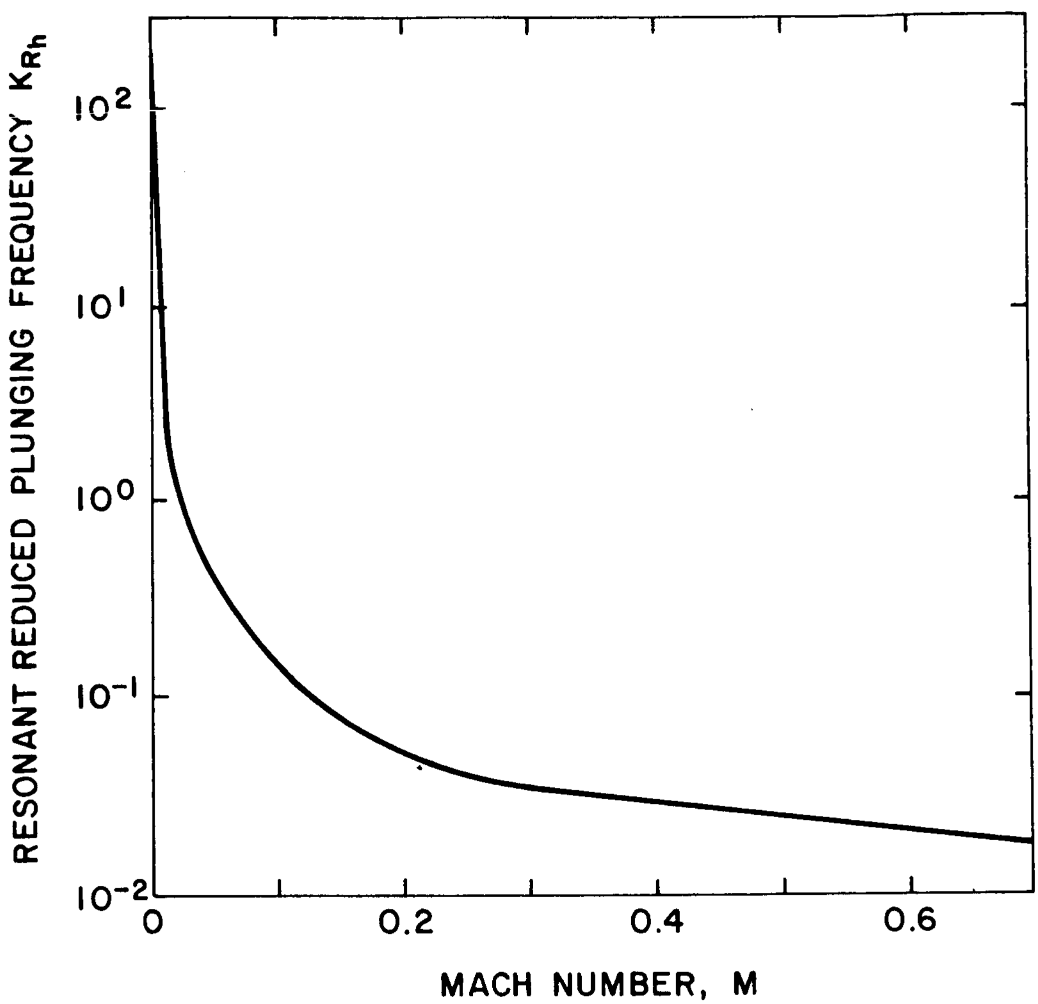

Figure 5.9 Variation of Resonant Plunging Reduced Frequency with Mach Number

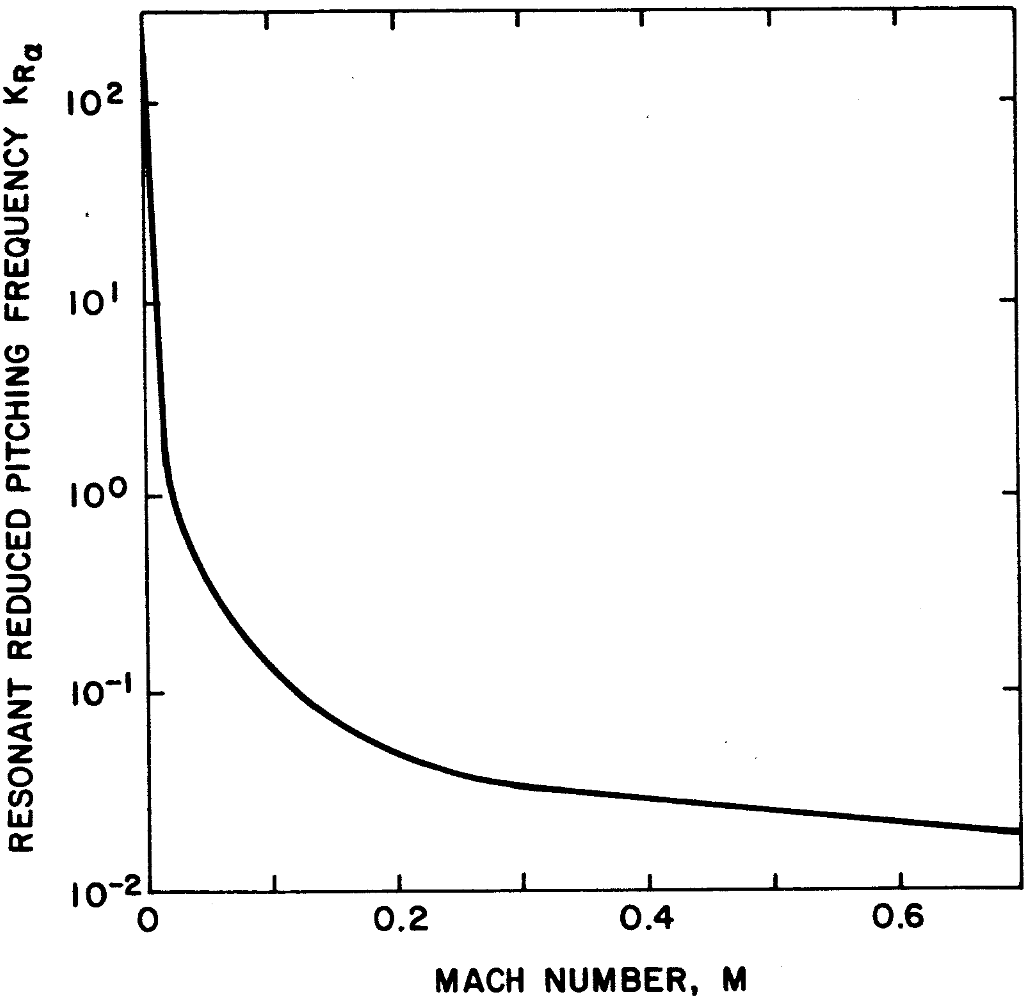

Figure 5.10 Variation of Resonant Pitching Reduced Frequency with Mach Number

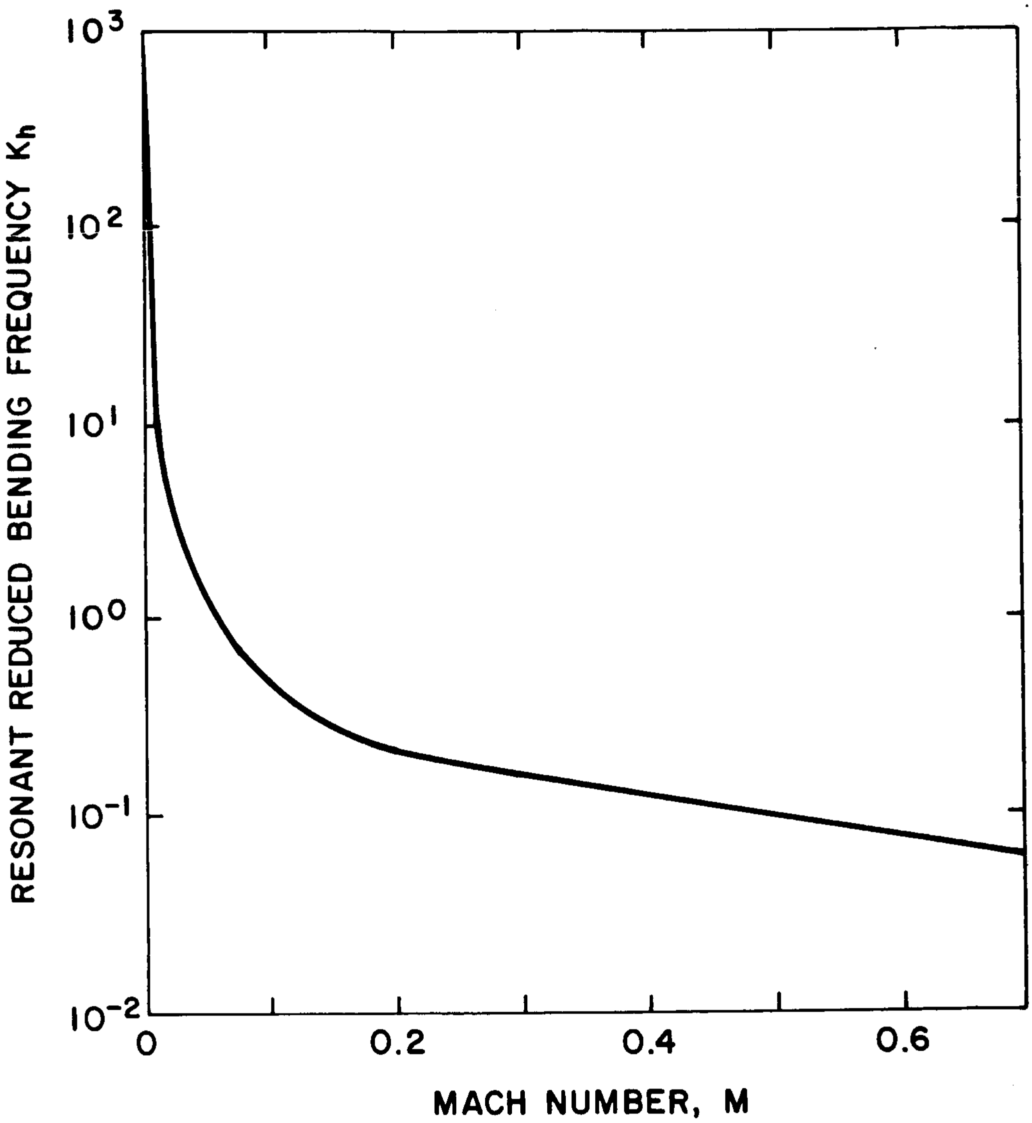

Figure 5.11 Variation of Resonant Bending Reduced Frequency with Mach Number

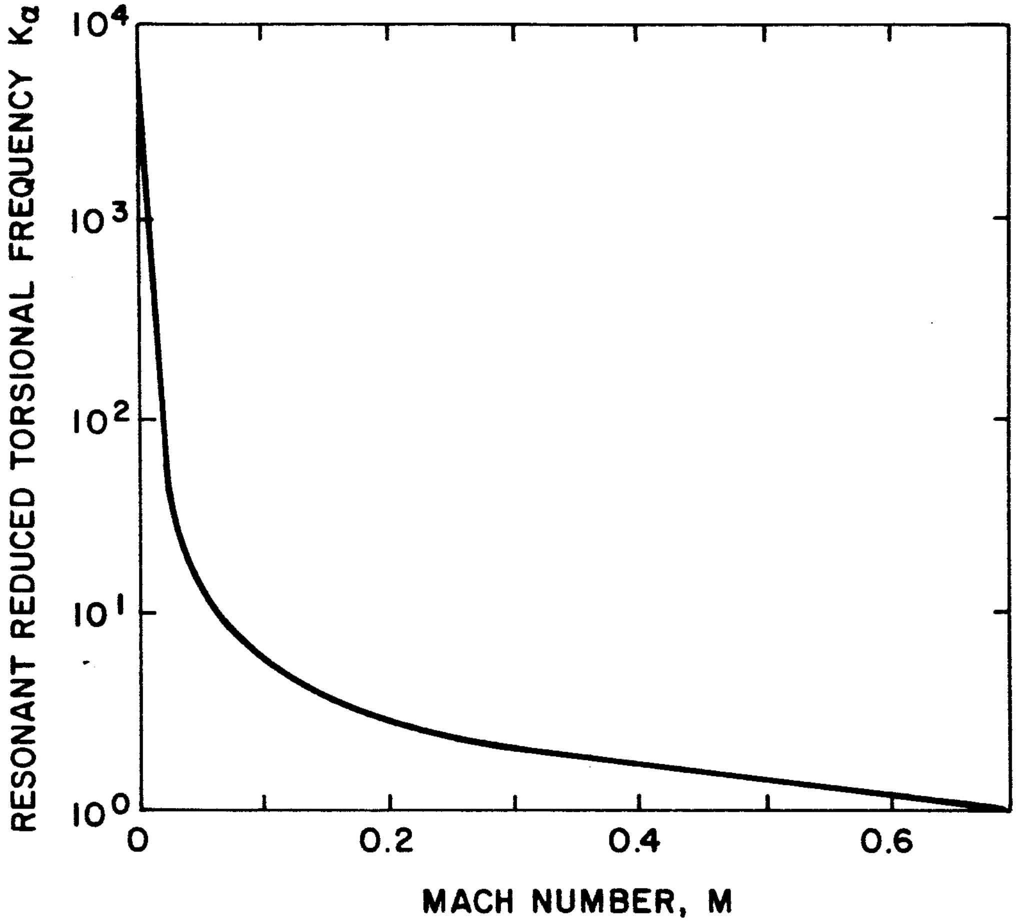

Figure 5.12 Variation of Resonant Torsional Reduced Frequency with Mach Number

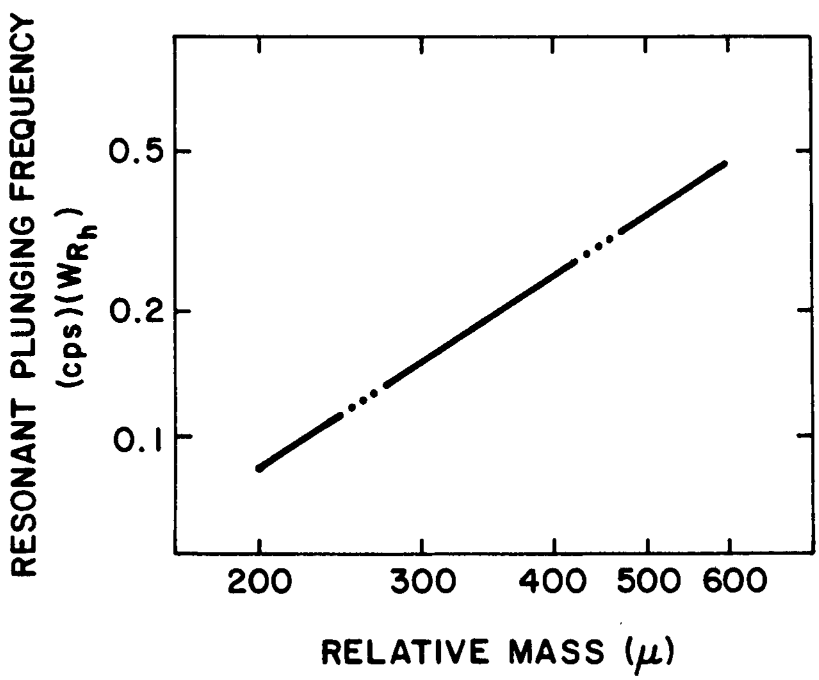

Figure 5.13 Variation of Resonant Circular Frequency (ωR_h) with Relative Mass

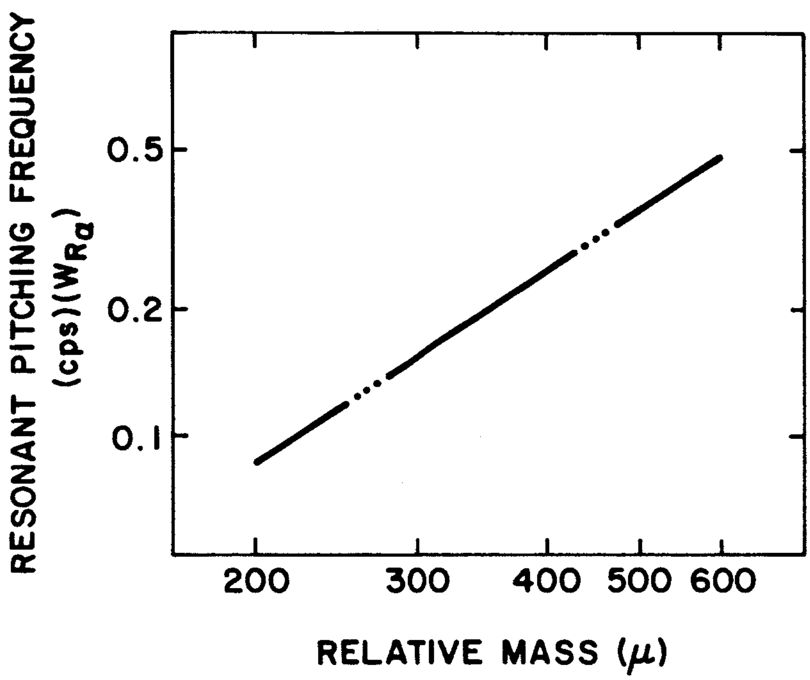

Figure 5.14 Variation of Resonant Circular Frequency (ωR_{α}) with Relative Mass

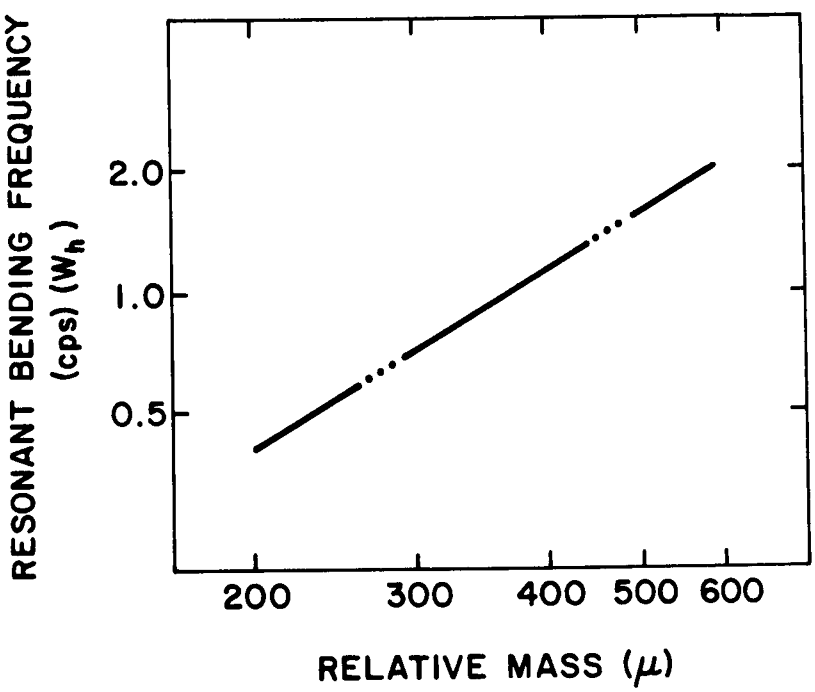

Figure 5.15 Variation of Resonant Circular Frequency (ω_h) with Relative Mass

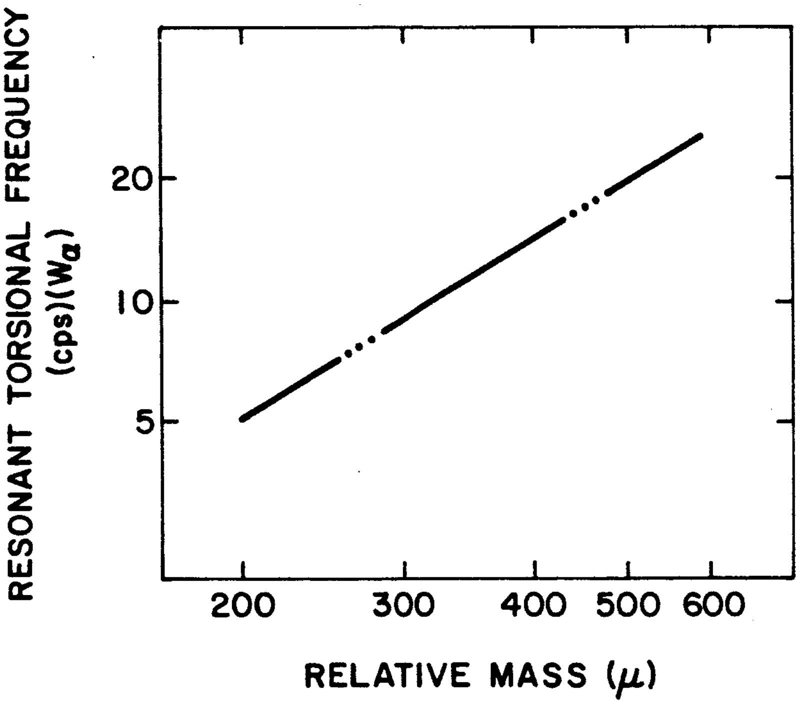

Figure 5.16 Variation of Resonant Circular Frequency (ω_α) with Relative Mass

These values of relative mass represent the extremes. Similarly, the torsional mode mechanical admittance decreases through this range to approximately one fourth of its value at $\mu = 200$. These decreases occur in a linear manner and are depicted in Figures 5.17 through 5.32.

The behavior of the mechanical admittance magnitudes with variations in Mach number may be seen in Figures 5.33 through 5.36. Contrary to experience, the rigid body modes are found to be particularly sensitive to Mach number variations. For this analysis, however, the controls have been assumed locked. This may account for the inconsistency between the numerical results and experience. The rigid body modes increase in magnitude as the Mach number approaches zero due to the longer duration gust. The decreasing magnitude of the cycle frequency tends to reduce the amplitude of the elastic modes. This accounts for the contrast between the rigid and elastic modes. The magnitudes shown in the figures represent only the resonant magnitudes.

5.5 Wavelength Critical Values for Resonant Response

A study by Ashburn, Waco, and Melvin [34], 1970, determined the wavelengths encountered in the high altitude clear air turbulence spectra. Root mean square gust velocities are established for the various values of wavelength and correlated to season of year, type of terrain, and altitude. It would be advantageous to determine those wavelength values which are critical to resonant response of an aircraft.

The circular frequency of encounter for these wavelengths may be expressed as,

$$\omega = \frac{2\pi\, u_o}{\bar{\lambda}} \qquad (5.10)$$

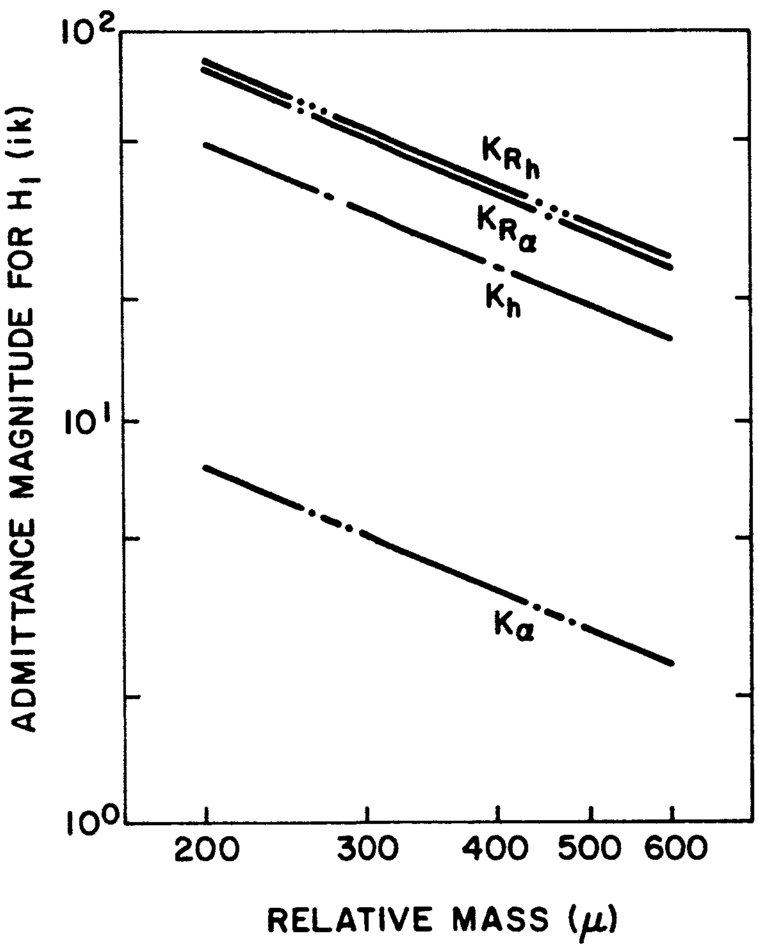

Figure 5.17 Variation of Admittance Magnitude $H_1(ik)$ with Relative Mass at Resonant Frequencies M = 0

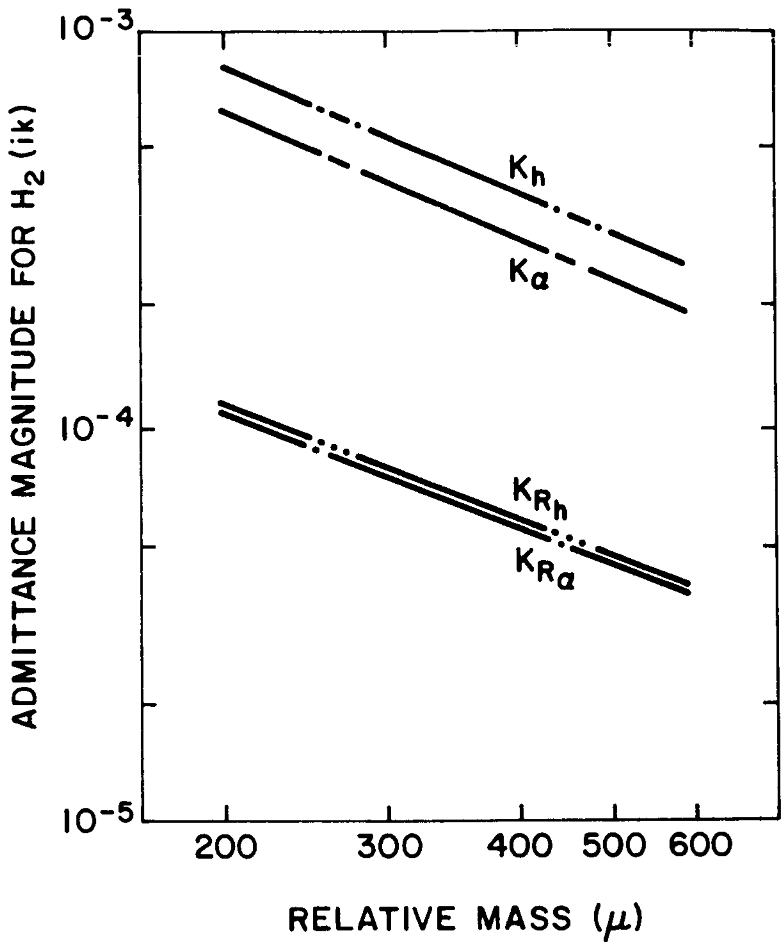

Figure 5.18 Variation of Admittance Magnitude $H_2(ik)$ with Relative Mass at Resonant Frequencies M = 0

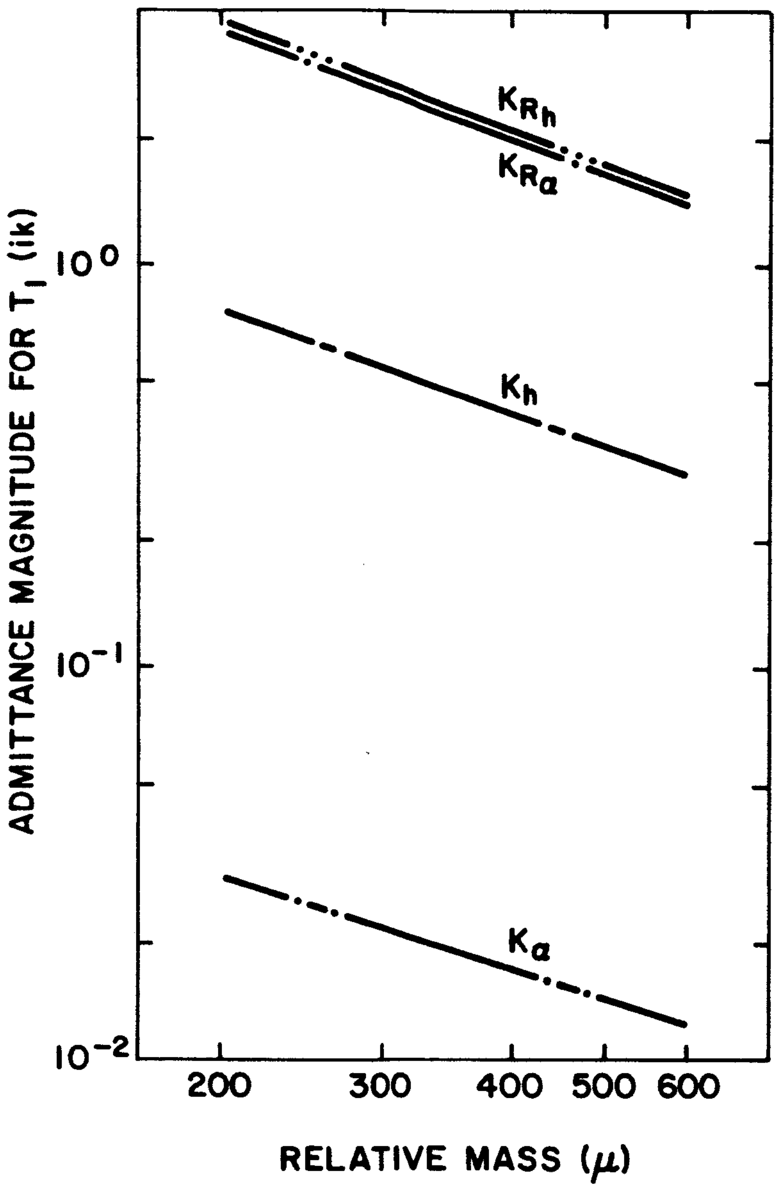

Figure 5.19 Variation of Admittance Magnitude $T_1(ik)$ with Relative Mass at Resonant Frequencies $M = 0$

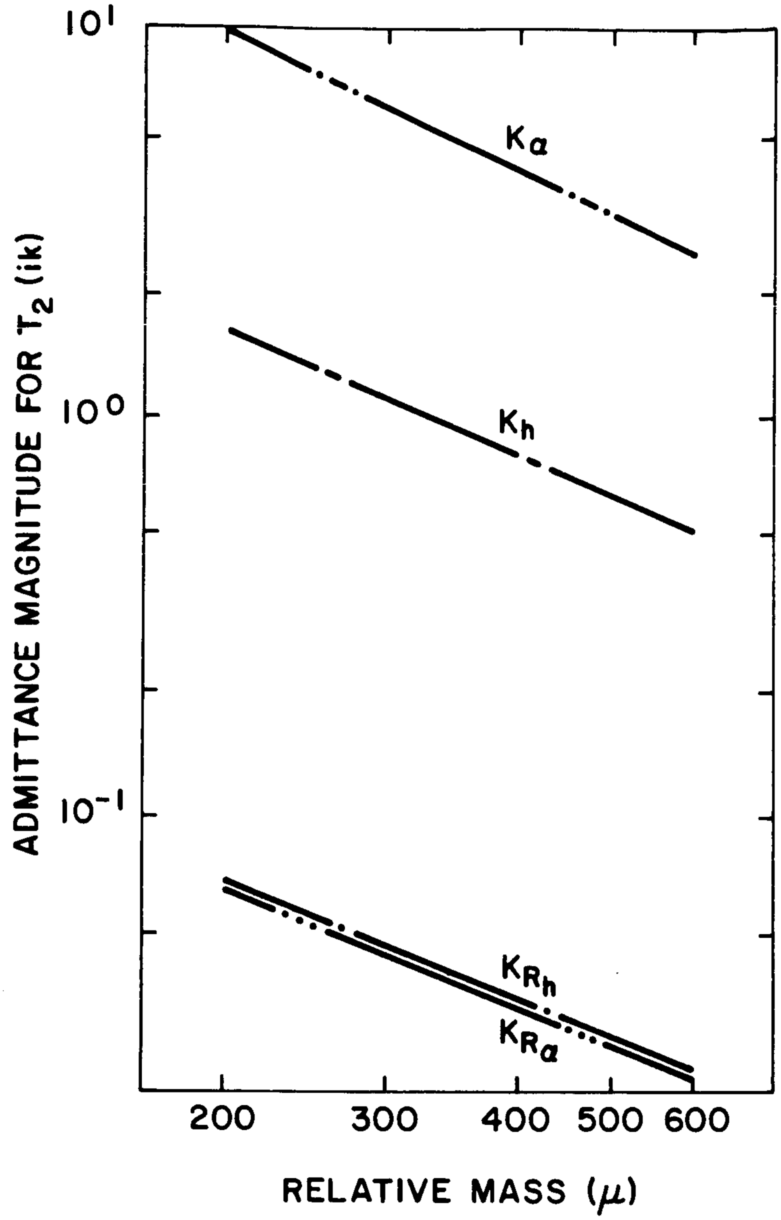

Figure 5.20 Variation of Admittance Magnitude $T_2(ik)$ with Relative Mass at Resonant Frequencies M = 0

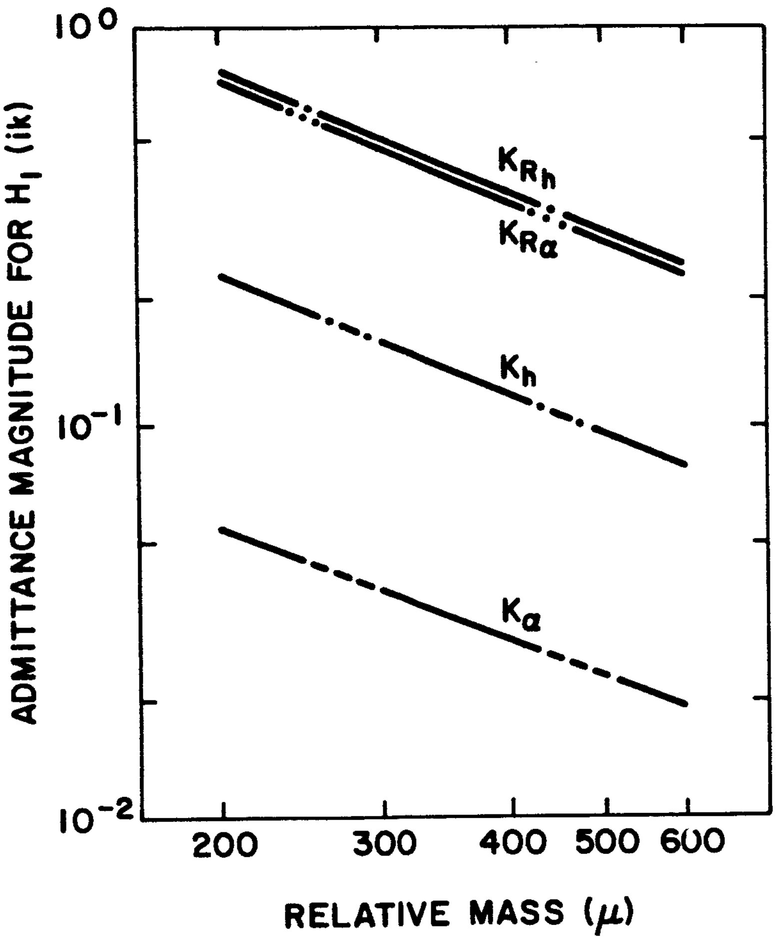

Figure 5.21 Variation of Admittance Magnitude H_1(ik) with Relative Mass at Resonant Frequencies M = 0.2

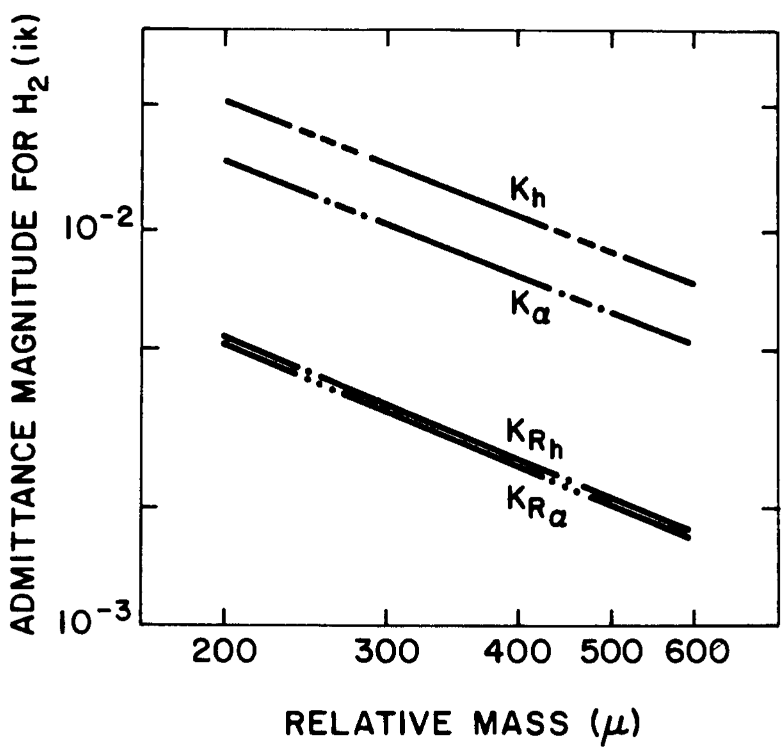

Figure 5.22 Variation of Admittance Magnitude H_2(ik) with Relative Mass at Resonant Frequencies M = 0.2

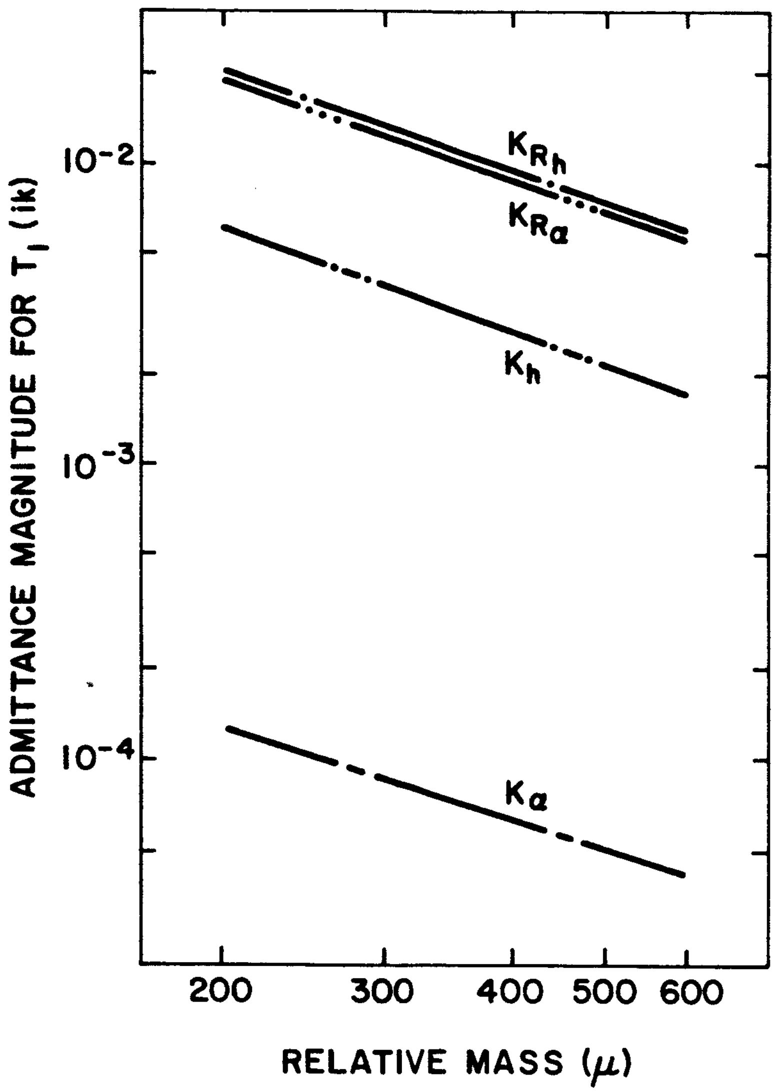

Figure 5.23 Variation of Admittance Magnitude T_1(ik) with Relative Mass at Resonant Frequencies $M = 0.2$

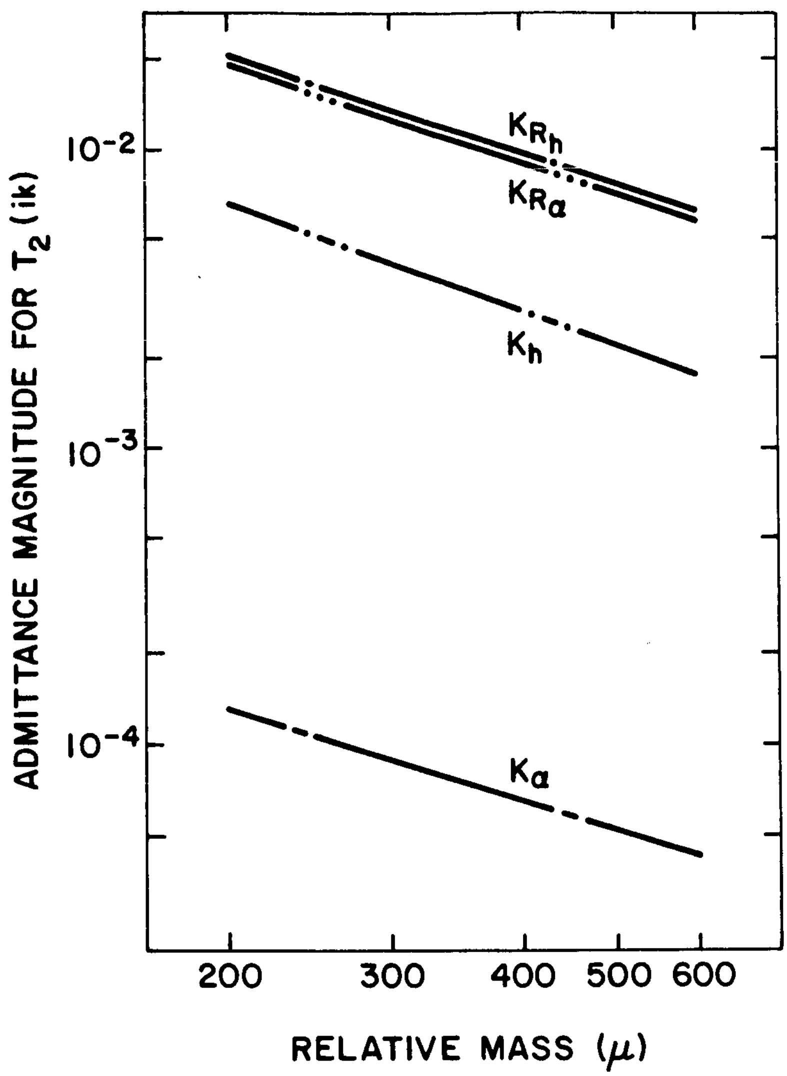

Figure 5.24 Variation of Admittance Magnitude T_2(ik) with Relative Mass at Resonant Frequencies M = 0.2

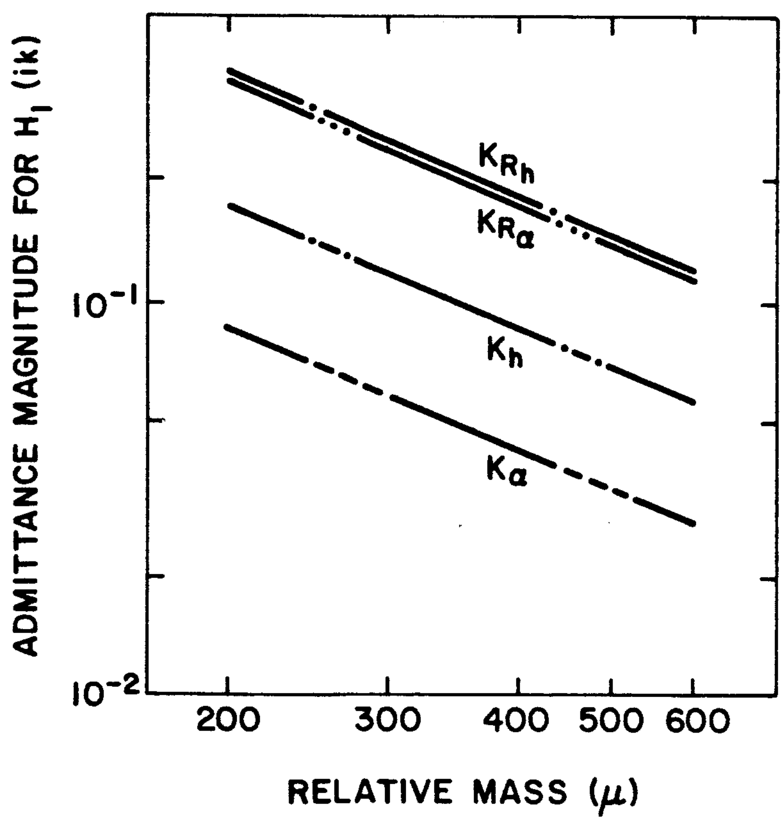

Figure 5.25 Variation of Admittance Magnitude H_1(ik) with Relative Mass at Resonant Frequencies M = 0.5

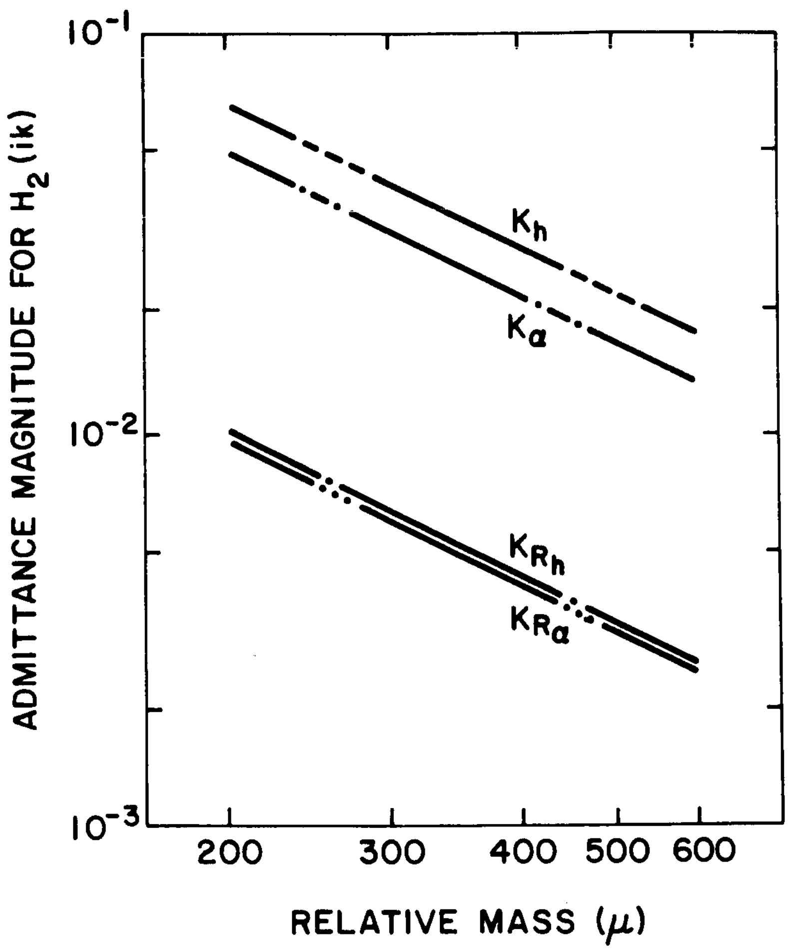

Figure 5.26 Variation of Admittance Magnitude $H_2(ik)$ with Relative Mass at Resonant Frequencies M = 0.5

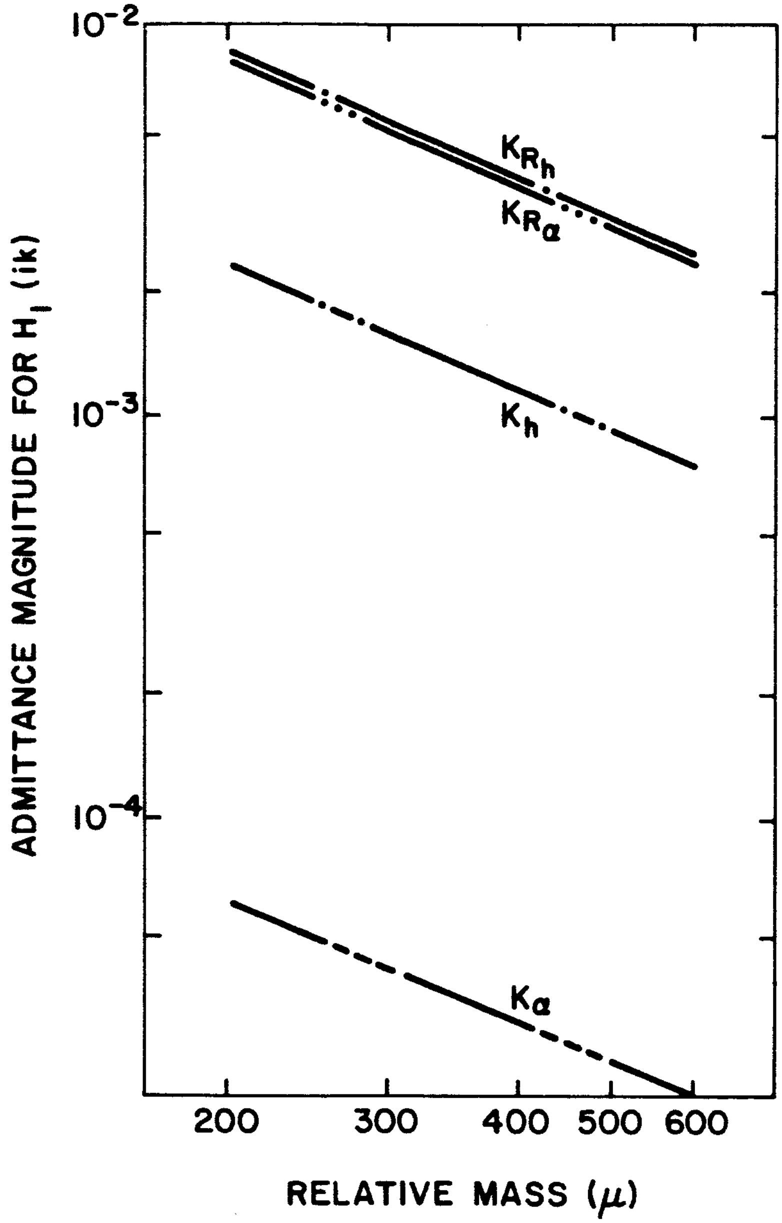

Figure 5.27 Variation of Admittance Magnitude $T_1(ik)$ with Relative Mass at Resonant Frequencies M = 0.5

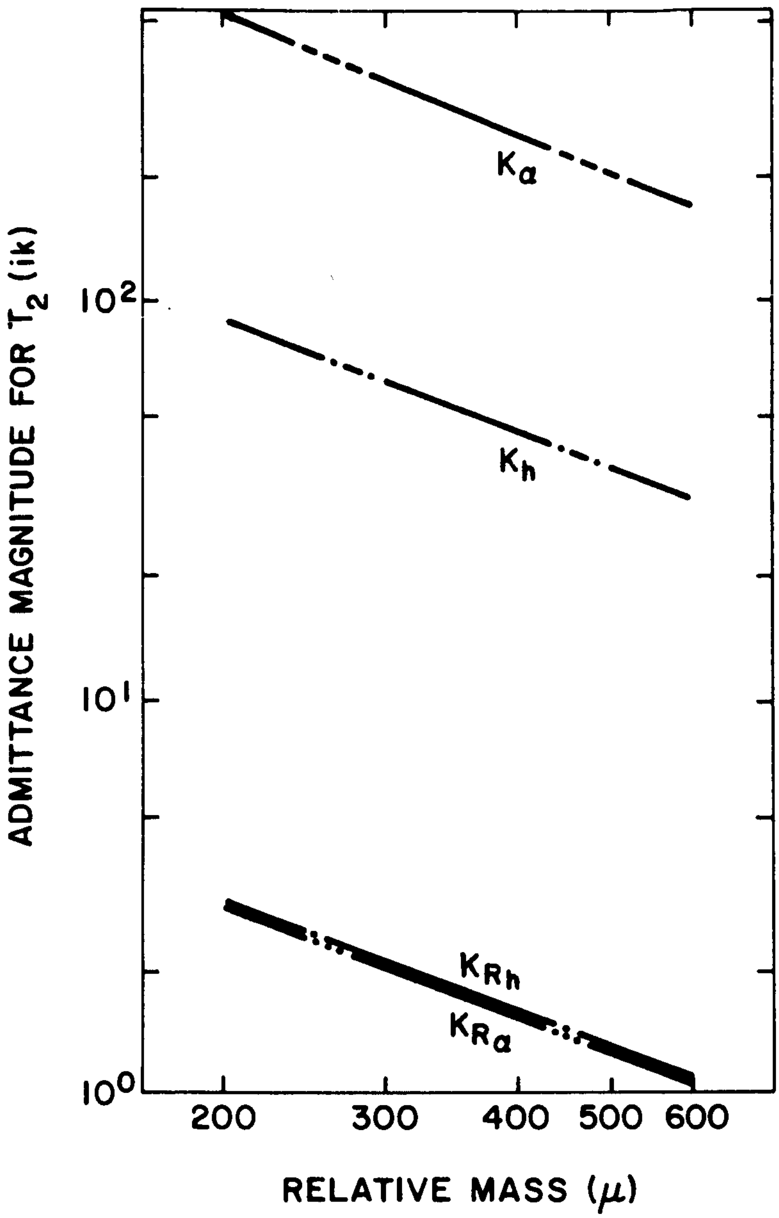

Figure 5.28 Variation of Admittance Magnitude T_2(ik) with Relative Mass at Resonant Frequencies M = 0.5

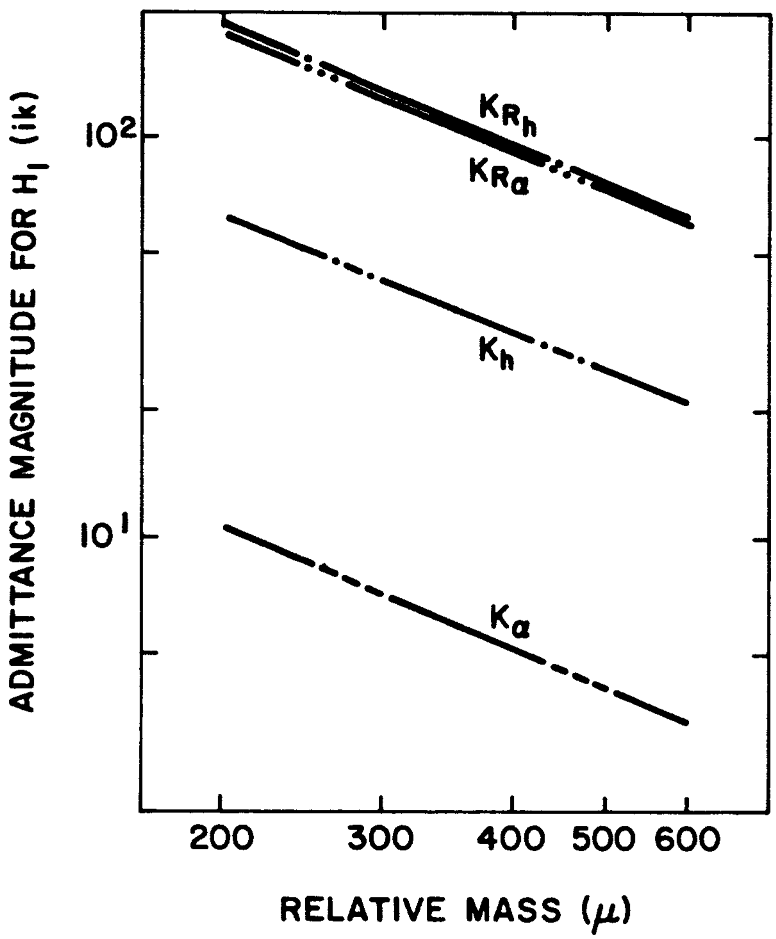

Figure 5.29 Variation of Admittance Magnitude $H_1(ik)$ with Relative Mass at Resonant Frequencies $M = 0.7$

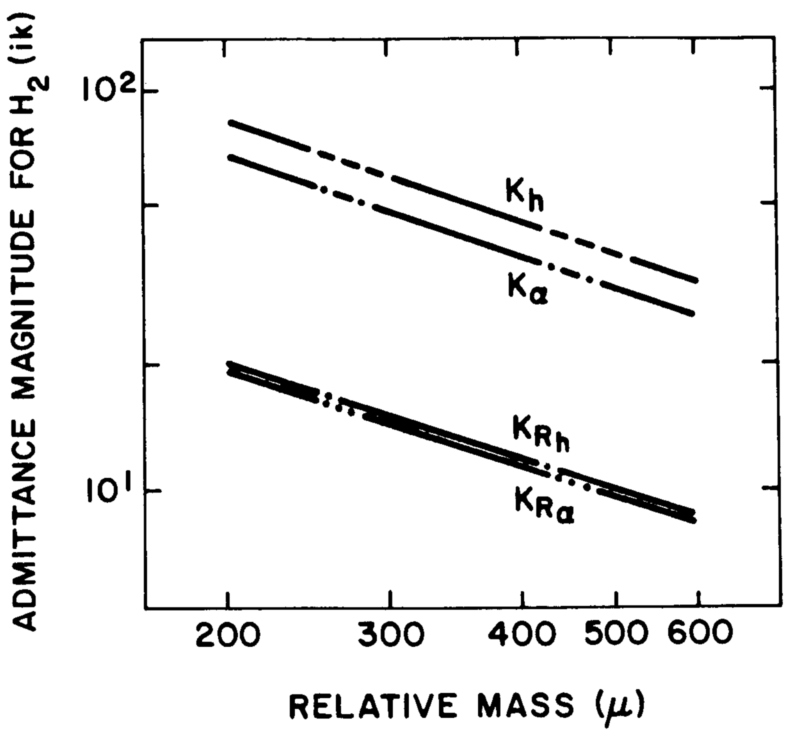

Figure 5.30 Variation of Admittance Magnitude $H_2(ik)$ with Relative Mass at Resonant Frequencies M = 0.7

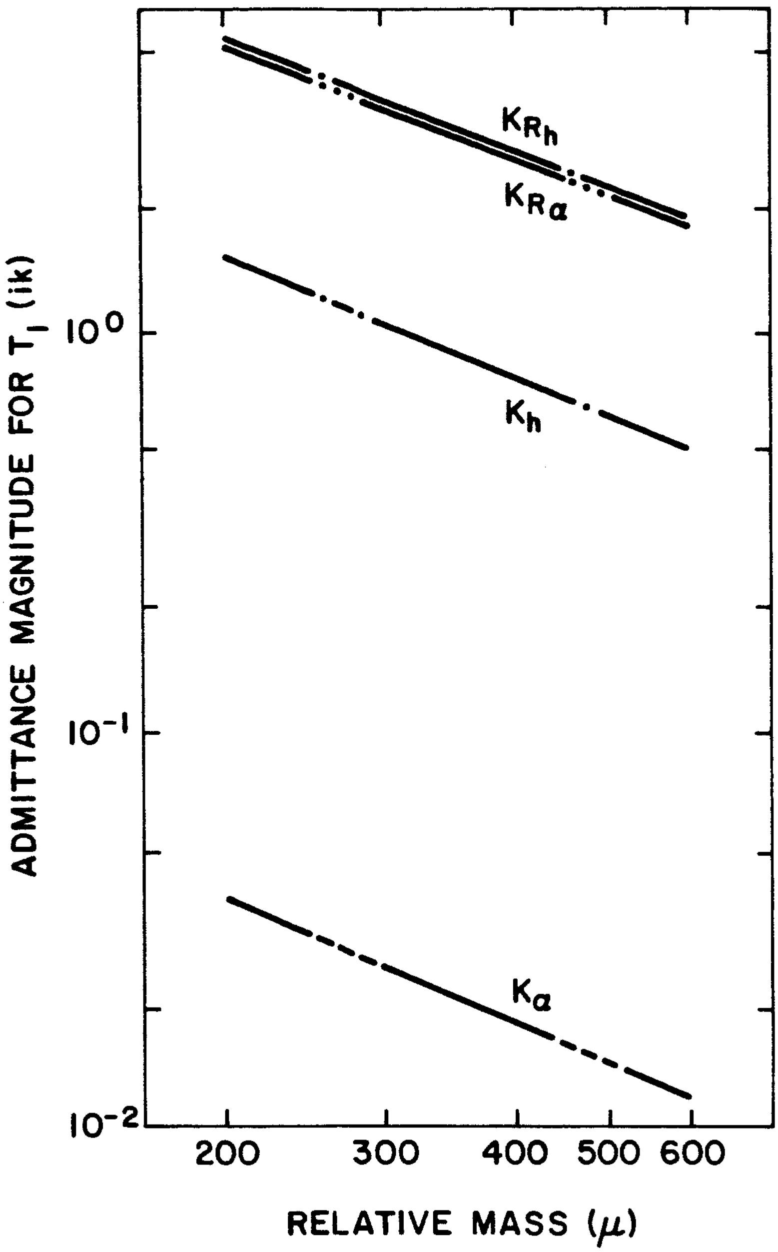

Figure 5.31 Variation of Admittance Magnitude T_1(ik) with Relative Mass at Resonant Frequencies M = 0.7

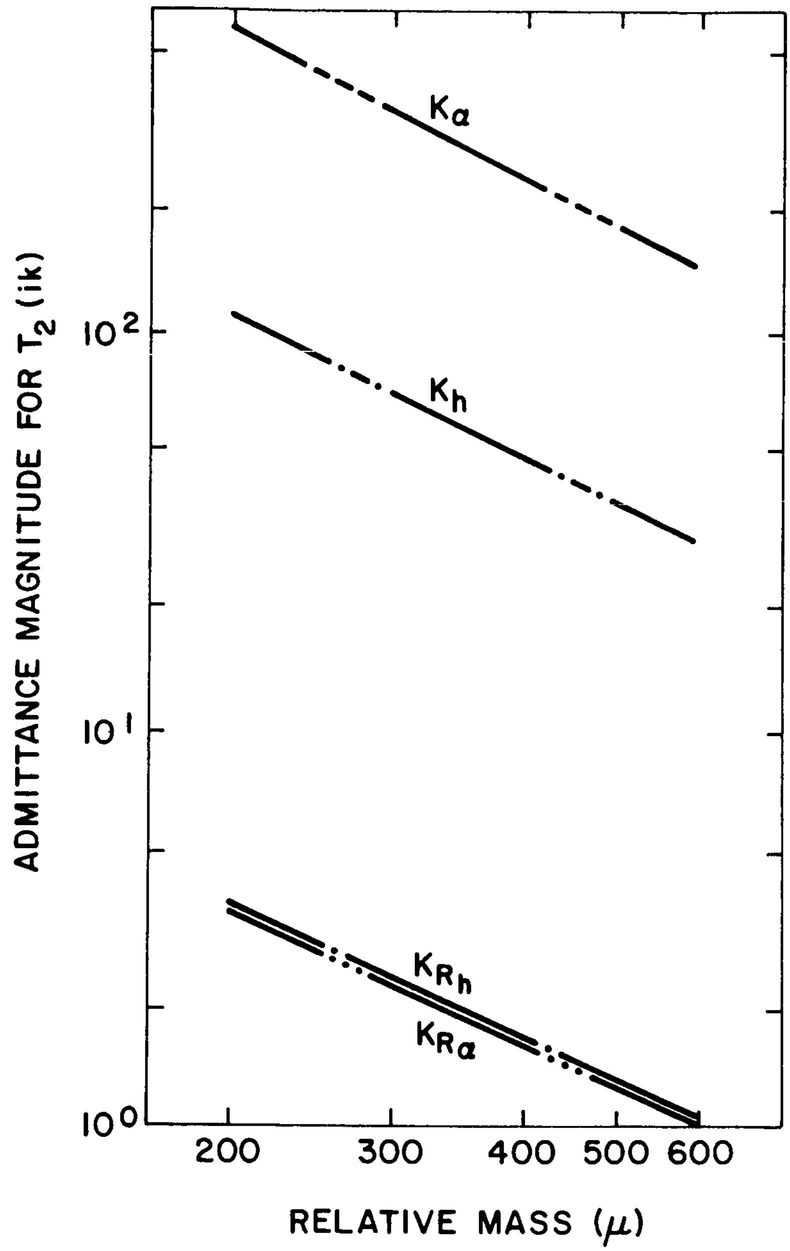

Figure 5.32 Variation of Admittance Magnitude T_2(ik) with Relative Mass at Resonant Frequencies M = 0.7

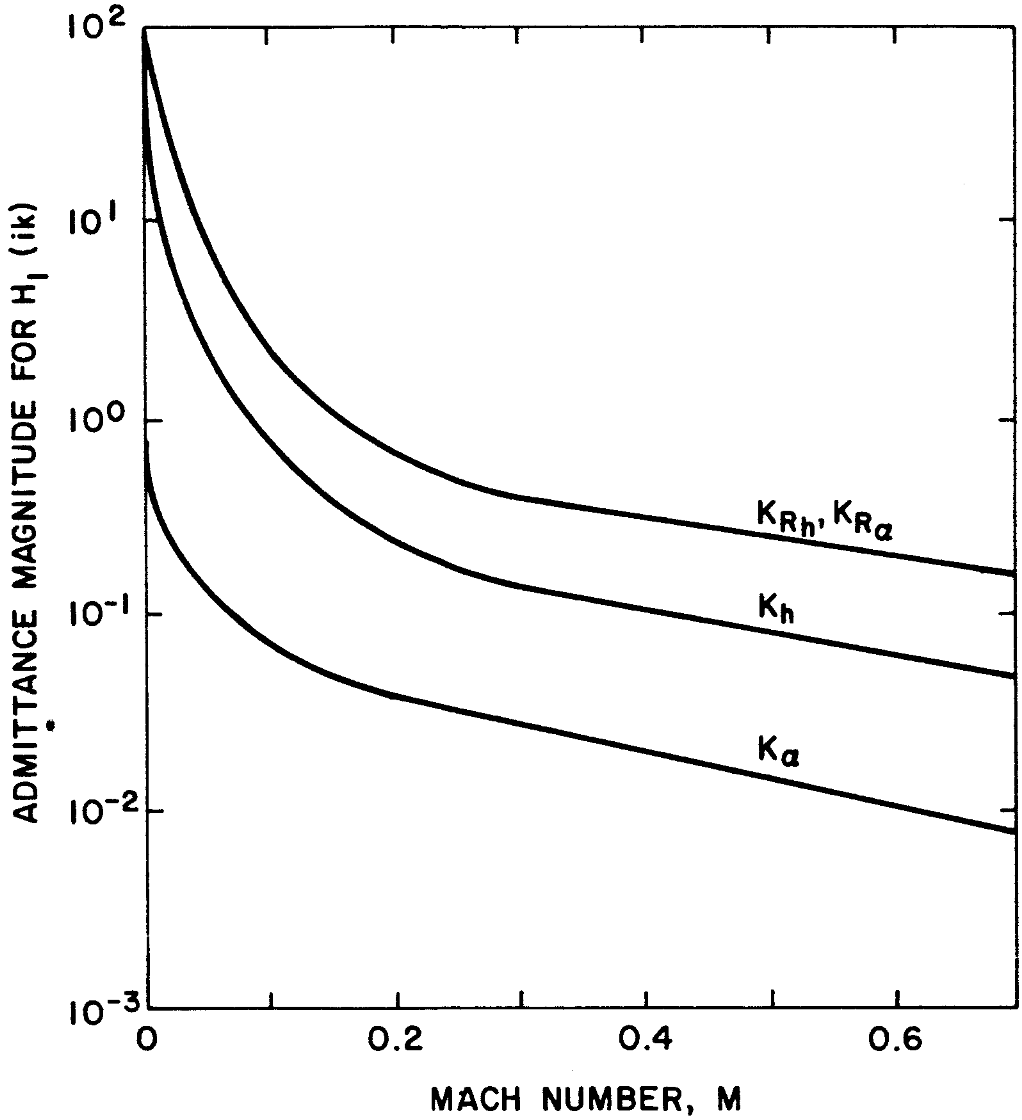

Figure 5.33 Variation of Admittance Magnitude H_1(ik) with Mach Number at Resonant Frequencies $\mu = 200$

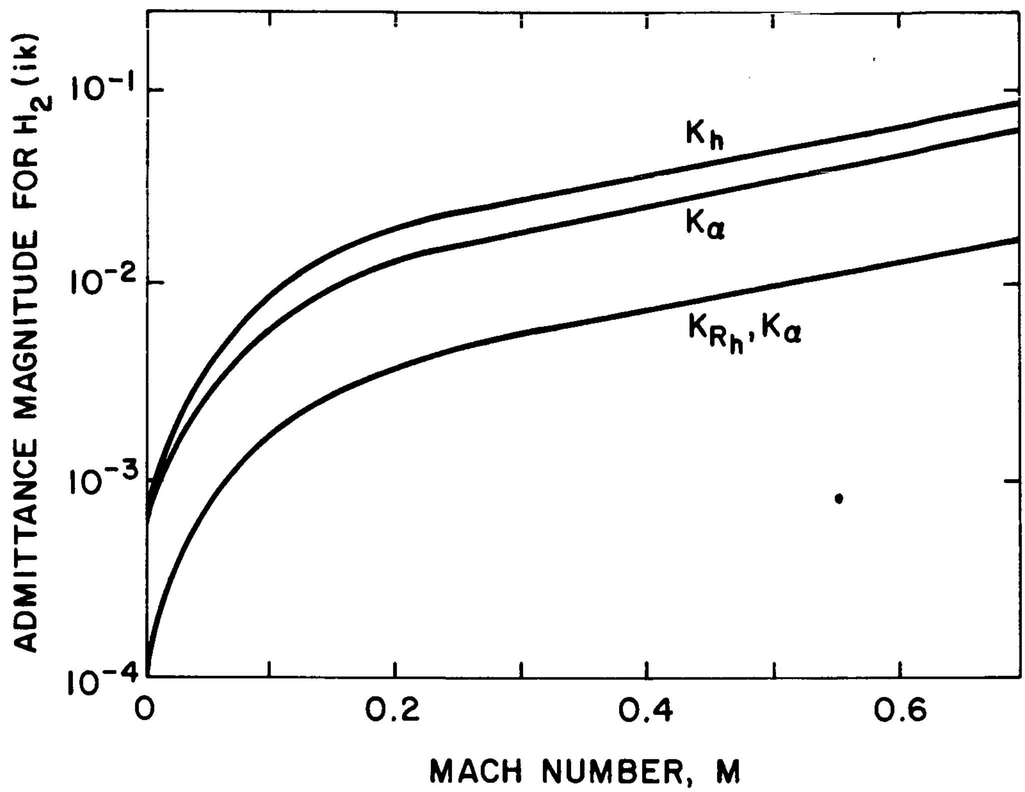

Figure 5.34 Variation of Admittance Magnitude H_2(ik) with Mach Number at Resonant Frequencies μ = 200

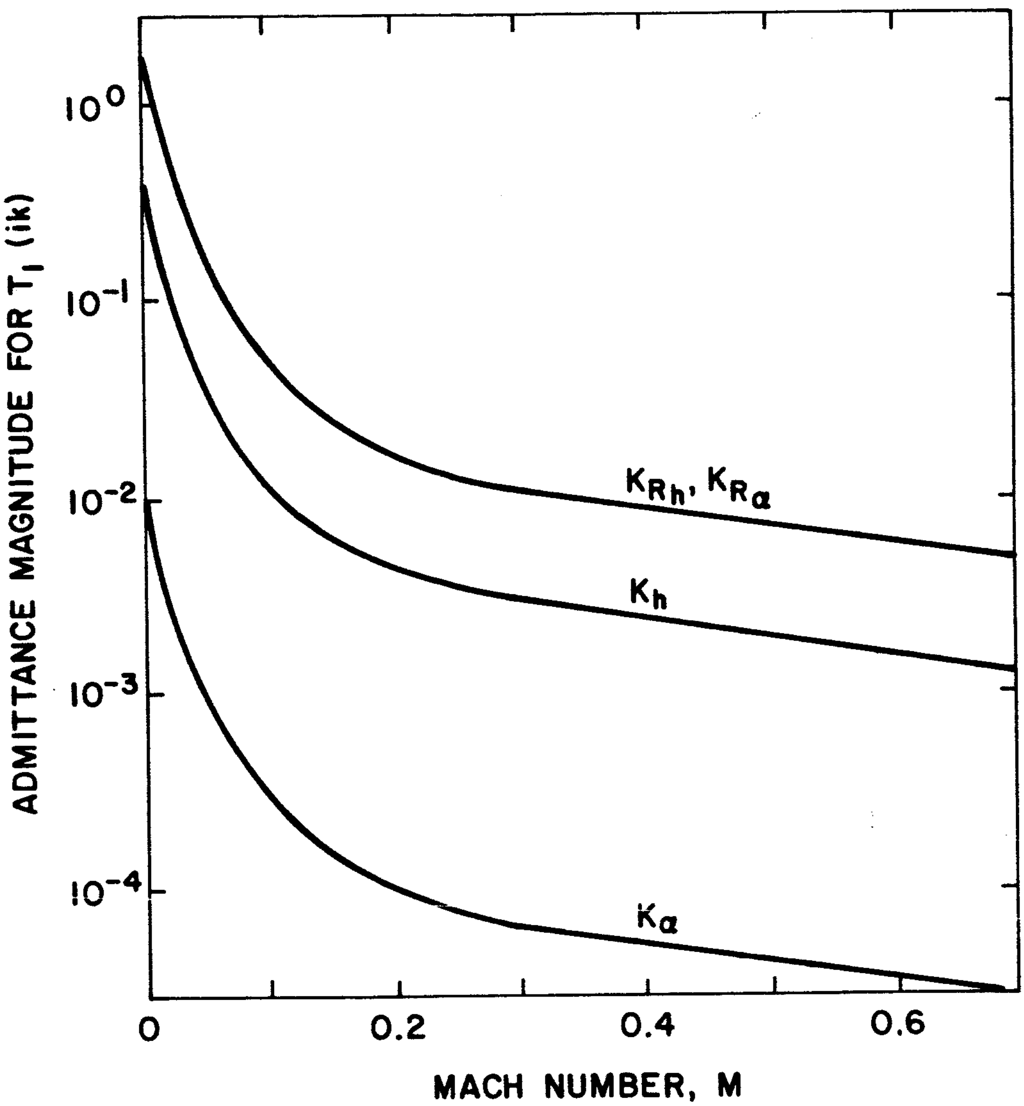

Figure 5.35 Variation of Admittance Magnitude $T_1(ik)$ with Mach Number at Resonant Frequencies $\mu = 200$

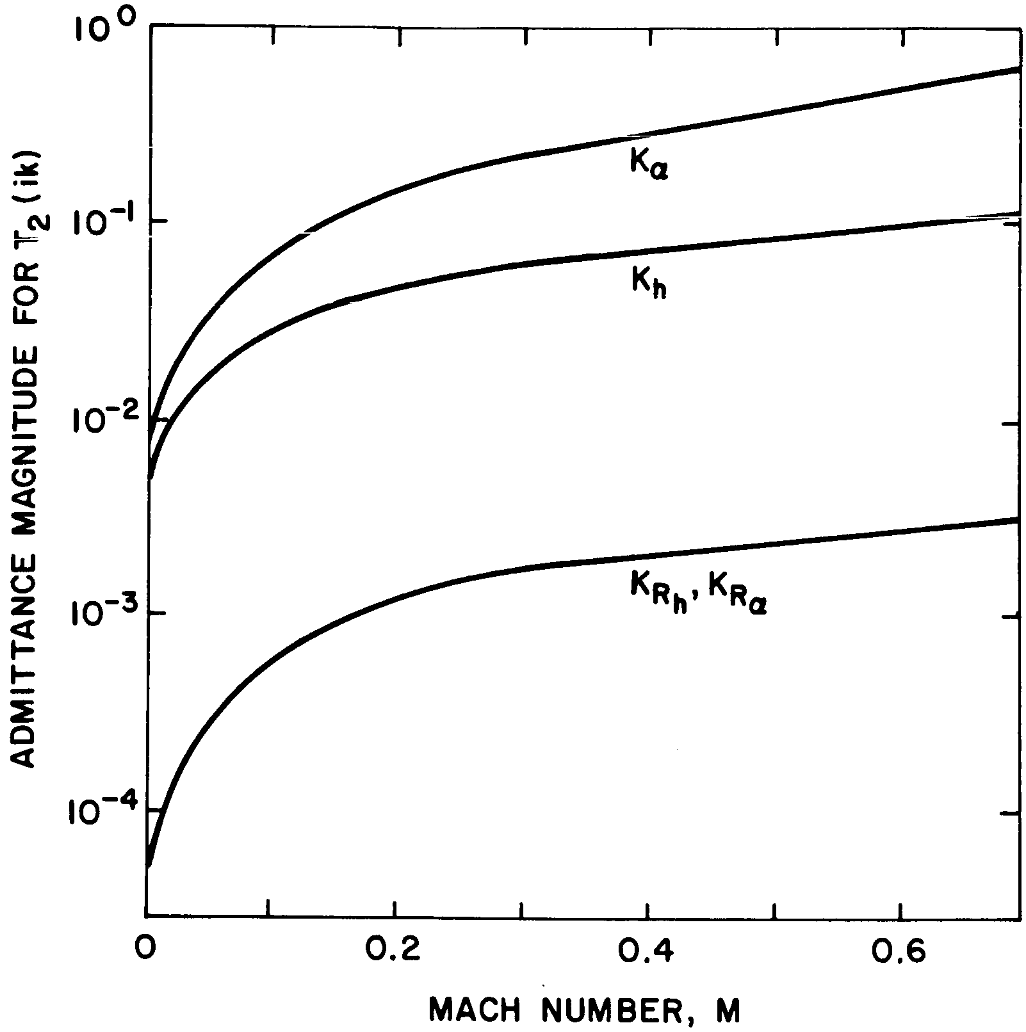

Figure 5.36 Variation of Admittance Magnitude T_2(ik) with Mach Number at Resonant Frequencies $\mu = 200$

The critical reduced forcing frequency may be expressed as

$$(5.11) \qquad k_c = \frac{\pi \bar{c}}{\bar{\lambda}}$$

Each mode considered in the analysis will have a critical wavelength which will be dependent on the mean aerodynamic chord $\bar{c}$, and the resonant reduced frequency. This value of wavelength is written:

$$(5.12) \qquad \bar{\lambda} = \frac{\pi \bar{c}}{k_c}$$

The resonant reduced frequencies were found to be independent of the relative mass parameter. Nondimensionalizing the resonant wavelength using the mean aerodynamic chord, the variation of the nondimensional wavelength with Mach number is presented in Figures 5.37 through 5.40.

The resonant wavelengths are plotted against relative mass of the craft, for Mach numbers equal to 0.2, 0.5, and 0.7, in Figures 5.41 through 5.43.

5.6 Conclusions

The following conclusions may be extracted from the results of this analysis.

(1) The response of an aircraft to atmospheric turbulence is dependent on the parameters of relative mass, Mach number, and the reduced frequency at which it is being forced.

(2) The ratio of mode reduced frequency parameter to forcing reduced frequency is dependent on Mach number and flight altitude.

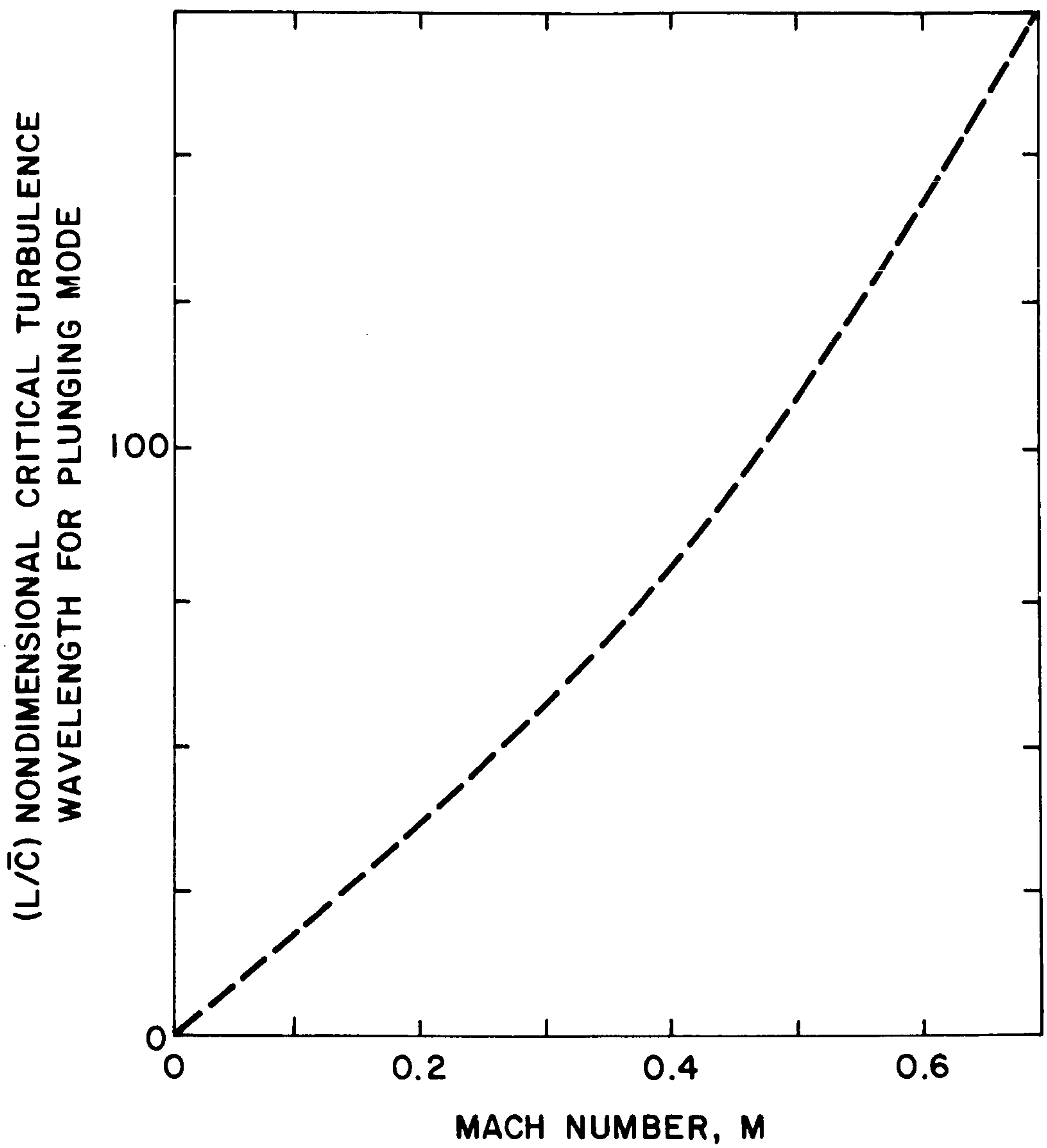

Figure 5.37 Variation of Nondimensional Critical Wavelength with Mach Number for Plunging Mode

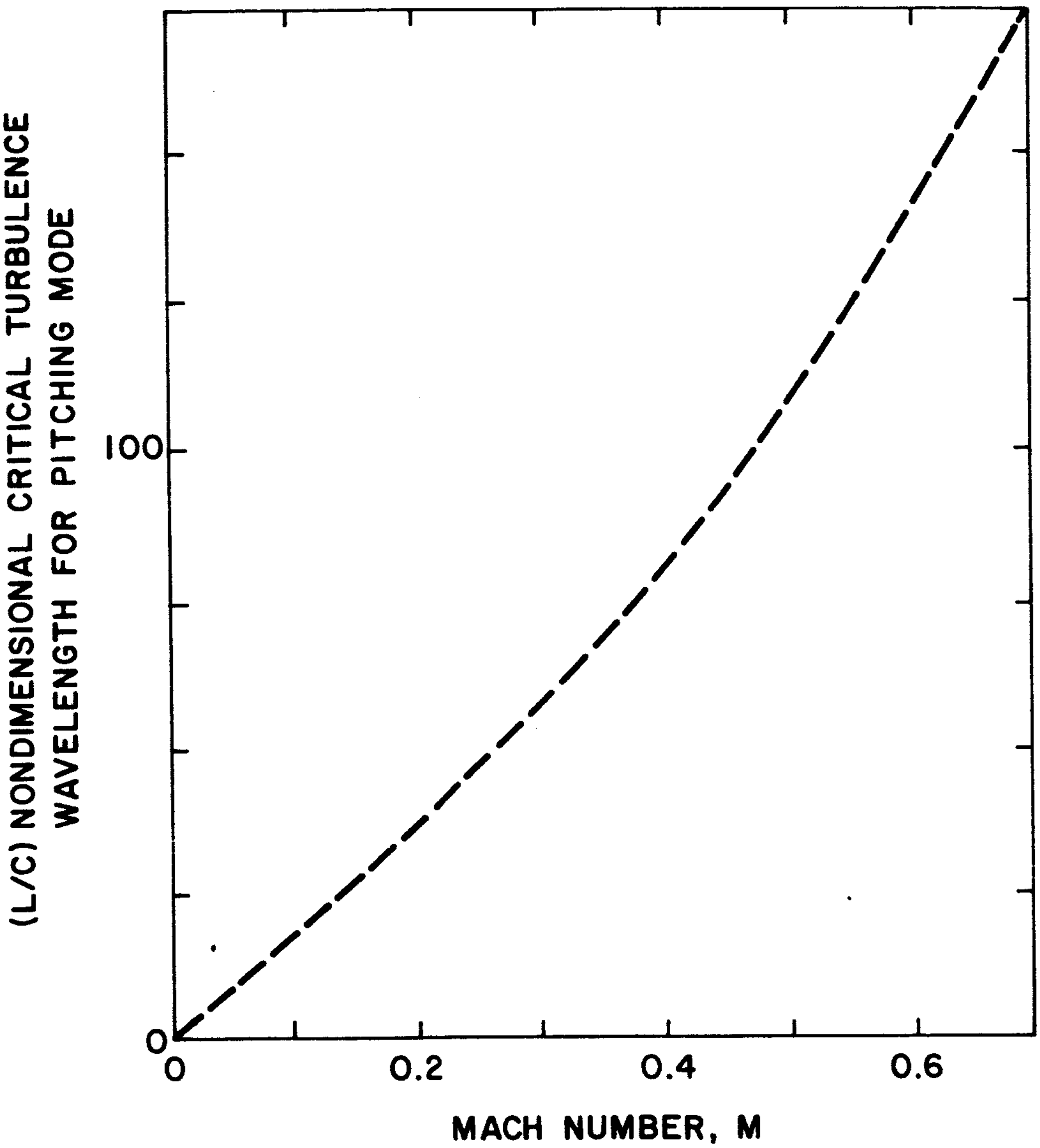

Figure 5.38 Variation of Nondimensional Critical Wavelength with Mach Number for Pitching Mode

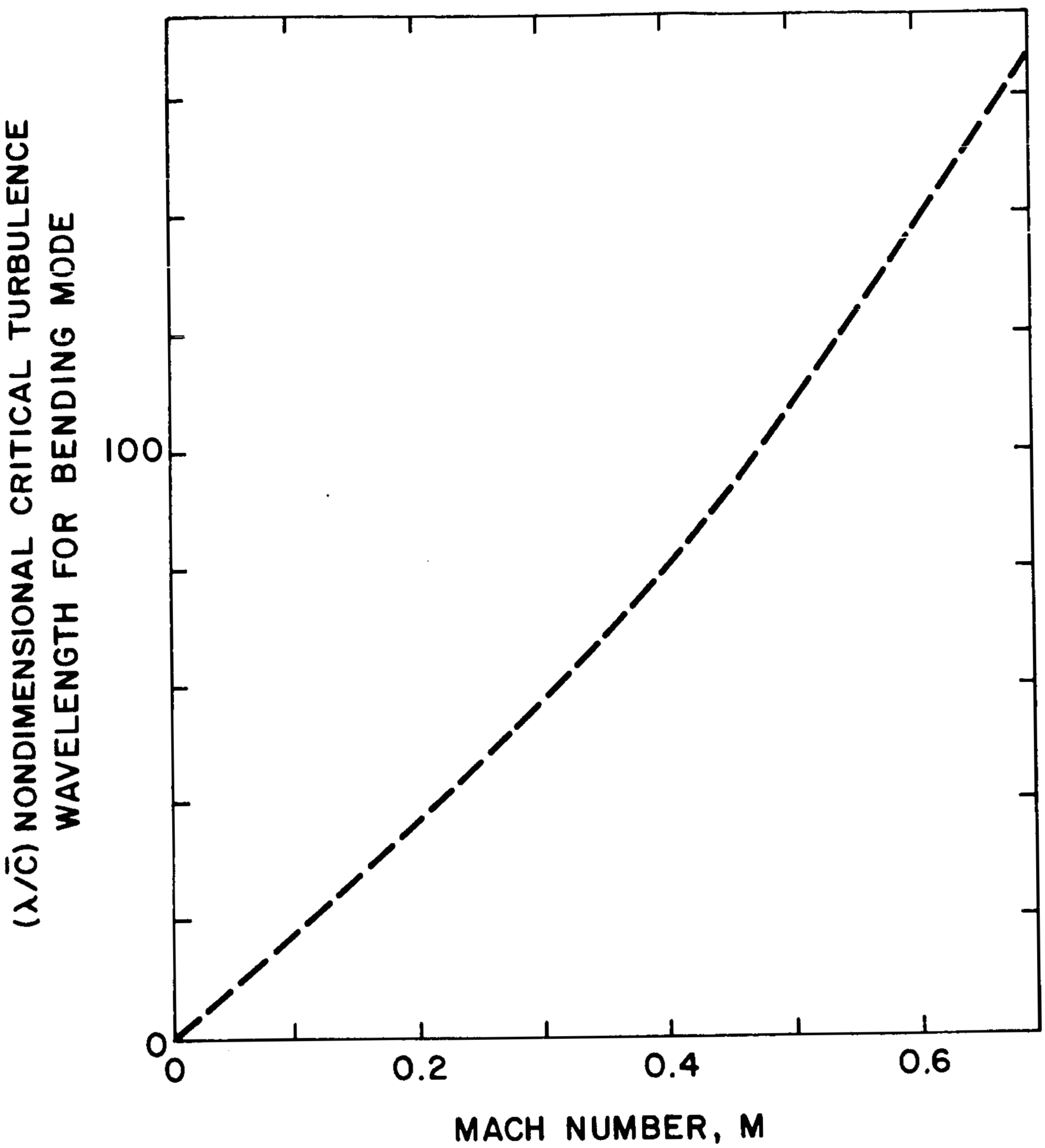

Figure 5.39 Variation of Nondimensional Critical Wavelength with Mach Number for Bending Mode

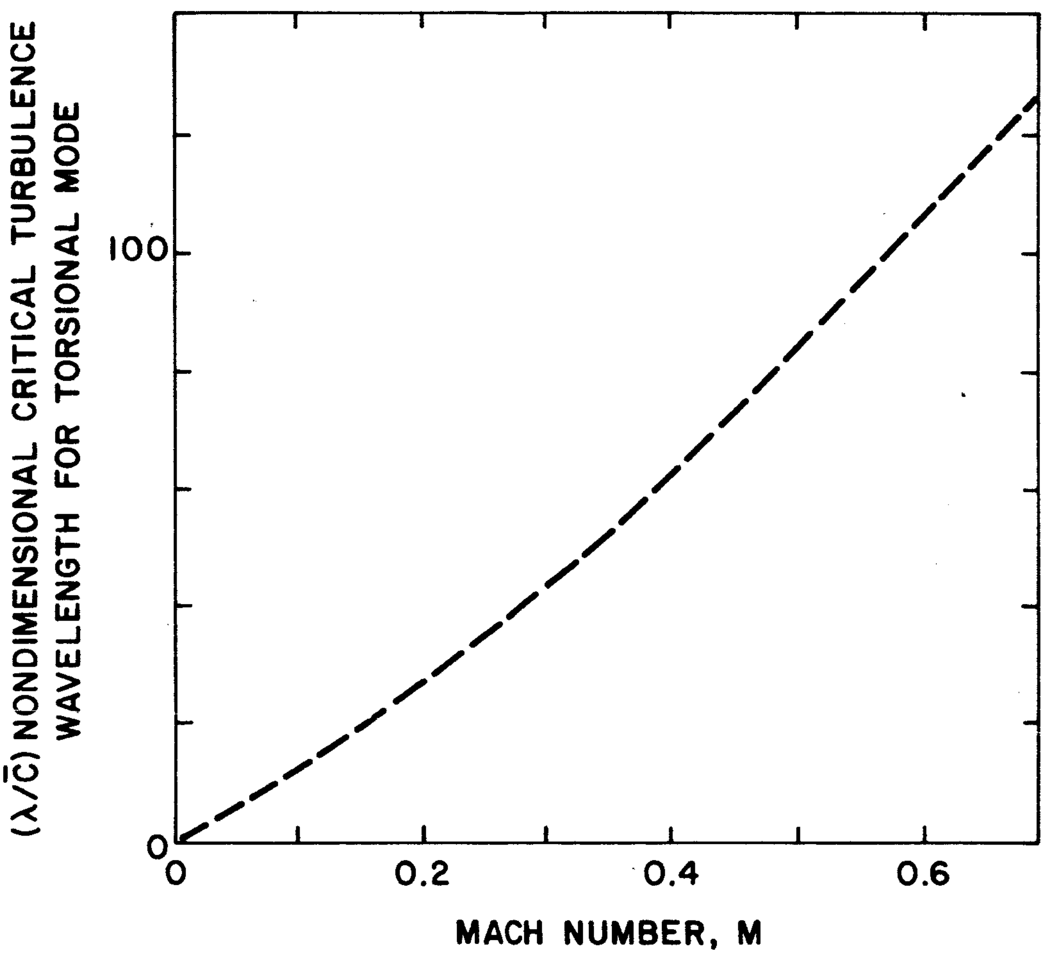

Figure 5.40 Variation of Nondimensional Critical Wavelength with Mach Number for Torsional Mode

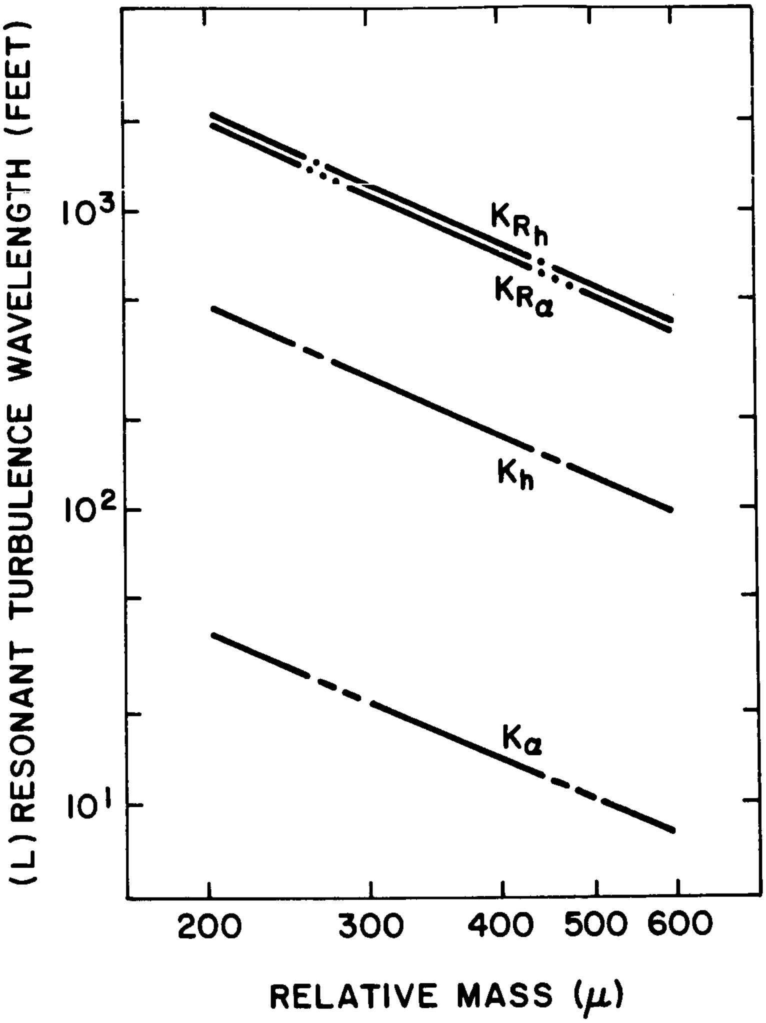

Figure 5.41 Resonant Turbulence Wavelength Variation with Relative Mass

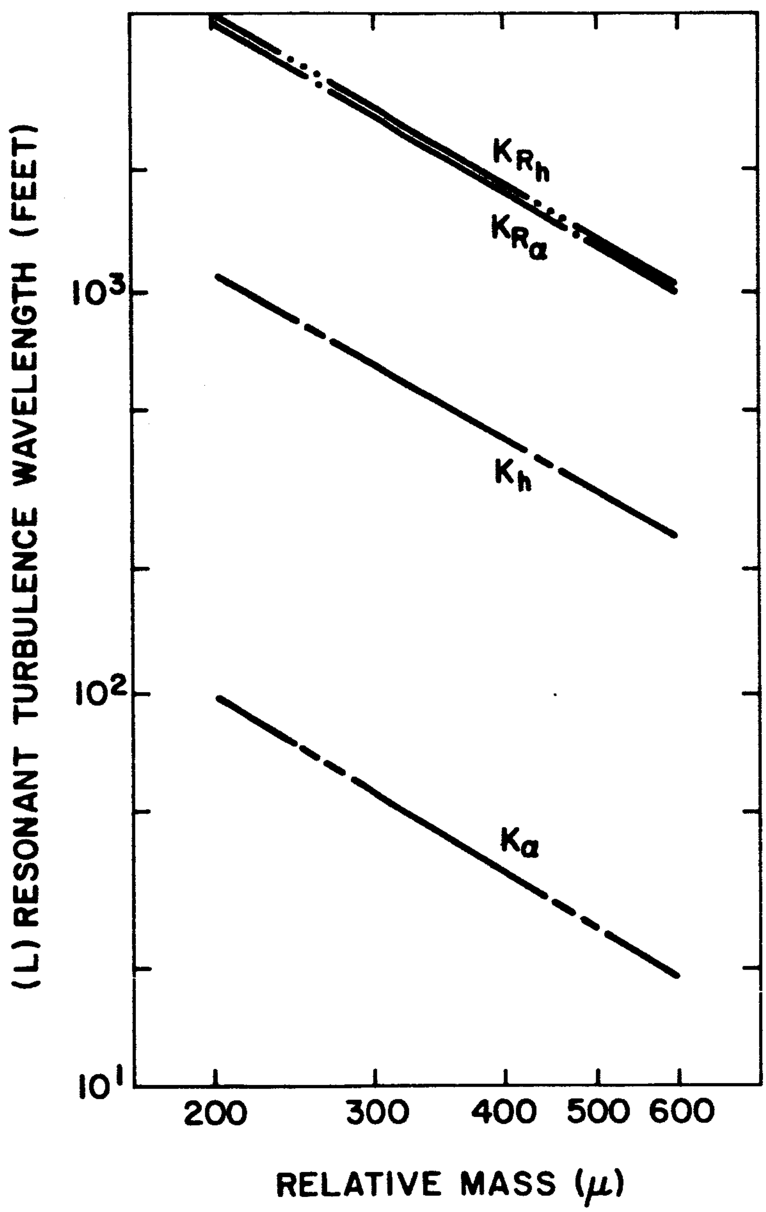

Figure 5.42 Resonant Turbulence Wavelength Variation with Relative Mass

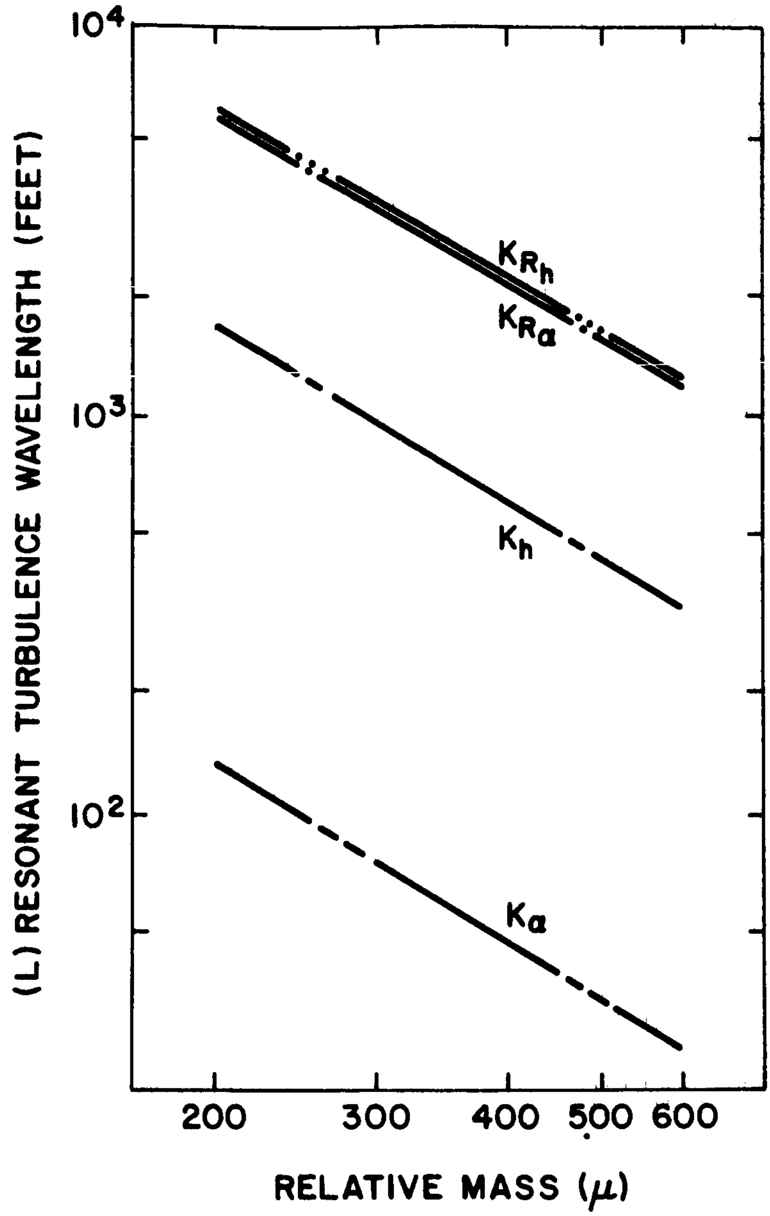

Figure 5.43 Resonant Turbulence Wavelength Variation with Relative Mass

(3) Secondary excitation of modes other than the mode under consideration will occur. The amplitude of secondary excitation can be comparable to the primary mode amplitude.

(4) Resonant frequencies remain nearly constant with variation in relative mass. These frequencies vary proportionally to the reciprocal of the Mach number at a given altitude.

(5) The magnitudes of the rigid modes increase with decreasing Mach number while the elastic modes decrease with decreasing Mach Number.

(6) The critical reduced frequencies are related to the reciprocal of the critical wavelengths of the turbulence spectra by the factor π times the mean aerodynamic chord.

(7) The critical wavelengths of the turbulence spectra decrease in a logarithmically linear fashion with logarithmically increasing relative mass.

5.7 Summary of Major Assumptions and Implications

(1) The aircraft was assumed to be traveling in steady level flight at an altitude of thirty thousand feet.

(2) Small disturbance approximations were made to linearize the response portion of the equations while the forcing terms were not linearized. This is a valid procedure as most disturbances do create response values of small magnitude.

(3) The wing of the craft was assumed to be elastic. The remainder of the craft was assumed rigid. This is a reasonable engineering approximation in considering only the pitching and plunging rigid modes and the first bending and torsional modes.

(4) Linear trends were established for wing and aircraft mass dependent and geometrical parameters. These trends were only established for subsonic jet transports and should be recomputed for other types of craft.

(5) The wing was considered to be unswept and untapered. The existence of skew and variable cross section will alter frequency characteristics and forcing properties; however, the error introduced does not nullify the qualitative information deduced from the results.

(6) Wing elastic characteristics were established by treating the wing as a cantilever. Again, this introduces some error in the mode shapes assumed, but is qualitatively a satisfactory engineering approximation.

(7) The turbulence was assumed to be statistically stationary. Only vertical gusts were considered.

5.8 Future Work

An extension of this method could lead to increased accuracy and further insight into the effect of several factors on the response of a craft to random gusts. The following considerations are suggested:

(1) The effects of spanwise variations of the unsteady aerodynamic loads on aircraft response.

(2) The use of three-dimensional unsteady aerodynamic theories.

(3) The incorporation of wing taper and sweepback.

(4) The inclusion of additional bending and torsional elastic modes for the wing.

(5) Consideration of other segments of the aircraft as elastic.

(6) The inclusion of chordwise gusts for analysis of the variations occurring in forward flight velocity. Lateral gusts might also be considered.

APPENDIX A

A COMPUTER PROGRAM -

NUMERICAL EVALUATION OF THE ADMITTANCE VALUES

A COMPUTER PROGRAM -

NUMERICAL EVALUATION OF THE ADMITTANCE VALUES

The numerical evaluation of the equations of motion may be accomplished through the use of the subroutine CLEQD which computes the solutions to a set of simultaneous linear equations with complex coefficients.

The following program has been utilized in order to allow the variation of the relative mass parameter through its realistic range in these equations.

```
    IMPLICIT REAL*8 (A,D-H, O-Z)
    COMPLEX*16 C(40,4,4),COM(4,4),B(4),BC(4),BET
    REAL*8 MU(40)
    DIMENSION A(4,4,4)
    WRITE(6,105)
105 FORMAT('1')
    XMU=200
    DMU=20
    DO 3 IROW=1,4
    DO 1 ICOL=1,4
    READ(5,100) (A(I,IROW,ICOL),I=1,4)
100 FORMAT(4(F12.7,8X))
  1 CONTINUE
    READ(5,102)B(IROW)
102 FORMAT(2F20.7)
  3 CONTINUE
    DO 4 IMU=1,40
    MU(IMU)=XMU
    DO 2 IROW=1,4
    DO 2 ICOL=1,4
    AX=A(1,IROW,ICOL)+A(2,IROW,ICOL)*XMU
    AY=A(3,IROW,ICOL)+A(4,IROW,ICOL)*XMU
    C(IMU,IROW,ICOL)=DCMPLX(AX,AY)
  2 CONTINUE
  4 XMU=XMU+DMU
    DO 5 IMU=1,40
    DO 6 IROW=1,4
    BC(IROW)=B(IROW)
    DO 6 ICOL=1,4
  6 COM(IROW, ICOL)=C(IMU,IROW,ICOL)
    CALL CLEQD(COM,BC,4,1,4,4,BET)
```

```
      WRITE(6,104) MU(IMU)
  104 FORMAT('-',50X,'MU= ',F10.5/)
      WRITE(6,103) (BC(IROW),IROW=1,4)
  103 FORMAT('Answer=',5X,D20.7,5X,D20.7,'*I')
    5 CONTINUE
      STOP
      END
```

APPENDIX B

MECHANICAL ADMITTANCE COMPONENTS

MECHANICAL ADMITTANCE COMPONENTS

The following tables list the real and imaginary components of the mechanical admittance. These are given for a variety of significant reduced frequency values and cover the entire range of relative mass for subsonic jet transports.

Table B.1 Mechanical Admittance Components

Reduced Frequency = 0.02 Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	.12716-02 .17798-02i	0.20736-04 0.22614-04i	-0.33522-04 0.37468-04i	0.98736-06 0.51647-07i
300	0.83774-03 0.13402-02i	0.22817-04 0.17632-04i	-0.26149-04 0.24729-04i	0.64337-06 0.30849-07i
400	0.55061-03 0.11019-02i	0.23035-04 0.13228-04i	-0.21792-04 0.17985-04i	0.44731-06 0.21634-07i
500	0.34713-03 0.90424-03i	0.21669-04 0.95173-05i	-0.20113-04 0.11990-04i	0.34768-06 0.12976-07i
600	0.22386-03 0.76003-03i	0.19893-04 0.68172-05i	-0.18453-04 0.93756-05i	0.27369-06 0.84316-08i

Reduced Frequency = 0.04 Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.63612-02 0.88993-02i	0.10513-03 0.12631-03i	-0.10385-03 -0.11858-03i	0.25642-04 0.12637-05i
300	-0.41908-02 0.67022-02i	0.11674-03 -0.96932-04i	-0.79252-04 -0.78543-04i	0.16849-04 -0.73942-06i
400	-0.27544-02 0.55087-02i	0.11939-03 -0.73694-04i	-0.66955-04 -0.56932-04i	0.11634-04 -0.50732-06i
500	-0.17366-02 0.45119-02i	0.10978-03 -0.51744-04i	-0.63001-04 -0.38217-04i	0.93746-05 -0.31769-06i
600	-0.11196-02 0.38001-02i	0.98210-04 -0.37422-04i	-0.57206-04 -0.29554-04i	0.72632-05 -0.31641-06i

TABLE B.1 (Continued)

Reduced Frequency = 0.06 Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.86702-01 0.12135-00i	0.67559-03 -0.74213-03i	-0.56950-03 -0.63750-03i	0.39671-03 -0.23617-04i
300	-0.57119-01 0.91386-01i	0.75211-03 -0.58614-03i	-0.44382-03 -0.41583-03i	0.26379-03 -0.14736-04i
400	-0.37542-01 0.75131-01i	0.76843-03 -0.45317-03i	-0.36855-03 -0.30621-03i	0.18237-03 -0.92632-05i
500	-0.23670-01 0.61528-01i	0.70966-03 -0.30127-03i	0.34208-03 -0.20483-03i	0.13614-03 -0.69361-05i
600	-0.15260-01 0.51894-01i	0.63715-03 -0.23144-03i	-0.15938-03i -0.15938-03i	-0.42734-05i -0.42734-05i

Reduced Frequency = 0.065 Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.40460-00 0.56630-00i	0.30475-02 0.33949-02i	-0.12067-01 -0.13506-01i	0.13228-02 -0.67336-04i
300	-0.26655-00 0.42648-00i	0.33742-02 -0.26881-02i	-0.93614-02 -0.89103-02i	0.86573-03 -0.45239-04i
400	-0.17531-00 0.35060-00i	0.34020-02 -0.21451-02i	-0.78074-02 -0.64859-02i	0.59376-03 -0.26933-04i
500	-0.11046-00 0.28711-00i	0.30893-02 -0.14622-02i	0.72341-02 -0.43214-02i	0.46321-03 -0.6572-04i
600	-0.71215-01 0.24181-00i	0.28316-02 -0.10194-02i	0.66012-02 -0.33757-02i	0.34270-03 -0.11747-04i

TABLE B.1 (Continued)

Reduced Frequency = 0.08 Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.63580-02 -0.88992-02i	0.92664-03 0.10649-02i	0.60355-03 -0.67531-03i	0.94765-04 0.51637-05i
300	-0.41888-02 -0.67020-02i	0.10436-02 0.82317-03i	0.47034-03 -0.44222-03i	0.65759-04 0.30842-05i
400	-0.27549-02 -0.55097-02i	0.10756-02 0.63187-03i	0.39195-03 -0.32649-03i	0.43662-04 0.20696-05i
500	-0.17357-02 -0.45119-02i	0.96037-03 0.43632-03i	0.36018-03 -0.21637-03i	0.34672-04 0.13264-05i
600	-0.11193-02 -0.38014-02i	0.89371-03 0.30026-03i	0.33165-03 -0.16875-03i	0.26312-04 0.91306-06i

Reduced Frequency = 0.10 Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.26588-02 0.37214-02i	0.72936-03 0.82642-03i	0.41372-03 -0.45376-03i	0.54639-04 -0.26394-05i
300	-0.17516-02 -0.28026-02i	0.80562-03 0.65147-03i	0.32659-03i -0.30744-03i	0.33947-04 0.16543-05i
400	-0.11521-02 -0.23039-02i	0.82043-03 0.50477-03i	0.28552-03 -0.23678-03i	0.24849-04 0.10637-05i
500	-0.72596-03 -0.18867-02i	0.76265-03 0.34717-03i	0.24479-03 -0.16137-03i	0.17650-04 0.62573-05i
600	-0.46797-03 -0.15891-02i	0.69459-03i 0.25742-03i	0.22745-03 -0.11632-03i	0.13705-04 -0.43671-06i

TABLE B.1 (Continued)

Reduced Frequency = 0.16 Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.15606-02 0.21843-02i	-0.55673-03 0.61575-03i	0.27564-03 0.30611-03i	0.36214-04 0.19735-05i
300	-0.10281-02 0.16450-02i	-0.61410-03 0.50731-03i	0.21745-03 0.20967-03i	0.24379-04 0.12598-05i
400	-0.67638-03 0.13523-02i	-0.62764-03 0.37659-03i	0.18421-03 0.15007-03i	0.16845-04 0.81732-06i
500	-0.42601-03 0.11074-02i	-0.60448-03 0.25397-03	0.16237-03 0.97536-04i	0.12073-04 0.50631-06i
600	-0.27017-03 0.93276-03i	-0.51243-03 0.19551-03i	0.14035-03i 0.82714-04i	0.10078-04 0.33649-06i

Reduced Frequency = 0.22 Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.23120-02 0.32363-02i	-0.13574-02 0.15173-02i	0.46592-03 0.53171-03i	0.99331-03 0.52997-04i
300	-0.15231-02 0.24373-02i	-0.14922-02 0.12363-02i	0.35256-03 0.38197-03i	0.65130-03 0.31746-04i
400	-0.10018-02 0.20036-02i	-0.15273-02 0.90455-03i	0.31778-03 0.27335-03i	0.44384-03 0.22374-04i
500	-0.63124-03 0.61409-02i	-0.14231-02 0.63712-03i	0.27849-03 0.17224-03i	0.35698-03 0.12634-04i
600	-0.40877-03 0.13819-02i	-0.12659-02 0.46128-03i	0.24639-03 0.13004-03i	0.29173-03 0.89312-05i

TABLE B.1 (Continued)

Reduced Frequency = 0.25 Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.14450-00 0.20225-00i	-0.14362-01 0.15873-01i	0.34785-03 0.37914-03i	0.62364-01 0.33419-02i
300	-0.95203-01 0.15231-00i	-0.15936-01 0.12964-01i	0.27145-03 0.25871-03i	0.41183-01 0.20445-02i
400	-0.62617-01 0.12521-00i	-0.16437-01 0.95489-02i	0.23369-03 0.19472-03i	0.27998-01 0.13256-02i
500	-0.39542-01 0.10254-00i	-0.14842-01 0.66765-02i	0.20843-03 0.13791-03i	0.20172-01 0.78463-03i
600	-0.25436-01 0.86361-01i	-0.13747-01 0.48172-02i	0.18954-03 0.96536-04i	0.16947-01 0.52366-03i

Reduced Frequency = 0.50 Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.57802-03 0.80936-03i	-0.24715-03 0.27442-03i	-0.58342-04 0.63772-04i	0.19973-04 0.10647-05i
300	-0.38080-03 -0.60953-03i	-0.27236-03 0.21547-03i	-0.46238-04 0.42693-04i	0.13695-04 0.63274-06i
400	-0.25046-03 -0.50107-03i	-0.27946-03 0.15884-03i	-0.39528-04 0.32147-04i	0.93672-05 0.43127-06i
500	-0.15780-03 -0.41035-03i	-0.28731-03 0.11221-03i	-0.35714-04i 0.21935-04i	0.72679-05 0.26334-06i
600	-0.10173-03 -0.34560-03i	-0.23514-03 0.82241-03i	-0.32142-04 0.16324-04i	0.61284-05 0.17379-06i

TABLE B.1 (Continued)

Reduced Frequency = 1.00 Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	0.28913-03 0.40450-03i	-0.12367-03 0.13649-03i	-0.23654-04 0.27136-04i	0.14115-04 0.73652-06i
300	0.19048-03 0.30463-03i	-0.13647-03 0.10748-03i	-0.18419-04 0.15937-04i	0.96813-05 0.43154-06i
400	0.12519-03 0.25043-03i	-0.13923-03 0.83651-04i	-0.16431-04 0.13613-04i	0.68317-05 0.30105-06i
500	0.78931-04 0.20508-03i	-0.12847-03 0.52761-04i	-0.14736-04 0.92371-05i	0.49377-05 0.18762-06i
600	0.50899-04 0.17272-03i	-0.11527-03 0.40761-04i	-0.11844-04 0.72779-05i	0.37146-05 0.12610-06i

Reduced Frequency = 2.00 Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.20231-03 -0.28315-03i	-0.87578-04 0.98273-04i	-0.16598-04 0.18846-04i	-0.12173-04 -0.63179-06i
300	-0.13328-03 -0.21324-03i	-0.96277-04 0.81448-04i	-0.12957-04 0.12647-04i	0.83654-05 -0.38762-06i
400	-0.87603-04 -0.17530-03i	-0.97863-04 0.59314-04i	-0.10751-04 0.94337-05i	0.54671-05 -0.25992-06i
500	-0.55234-04 -0.14356-03i	-0.91443-04 0.40879-04i	-0.98563-05 0.67336-05i	0.43612-05 -0.15837-06i
600	-0.35617-04 -0.12091-03i	-0.83842-04 0.28495-04i	-0.92634-05 0.45316-05i	0.33674-05 -0.10735-06i

TABLE B.1 (Continued)

Reduced Frequency = 2.50 Mechanical Admittance Magnitude M = 0.2

μ	H_1(ik)	H_2(ik)	T_1(ik)	T_2(ik)
200	-0.34682-03 -0.48540-03i	-0.93613-04 0.11375-03i	0.21640-04 -0.22703-04i	-0.12973-04 -0.66374-06i
300	-0.22849-03 -0.36555-03i	-0.10298-03 0.88784-04i	0.16327-04 -0.15216-04i	-0.87364-05 -0.40175-06i
400	-0.15028-03 -0.30051-03i	-0.10641-03 0.65943-04i	0.14733-04 -0.11536-04i	-0.59715-05 -0.26798-06i
500	-0.94683-03 -0.24610-03i	-0.97364-04 0.47331-04i	0.12373-04 -0.76534-05i	-0.43622-05 -0.16327-06i
600	-0.61045-03 -0.20727-03i	-0.89137-04 0.33752-04i	0.11470-04 -0.55236-05i	-0.13254-05 -0.11273-06i

Reduced Frequency = 3.00 Mechanical Admittance Magnitude M = 0.2

μ	H_1(ik)	H_2(ik)	T_1(ik)	T_2(ik)
200	-0.24276-02 -0.33978-02i	-0.18732-03 0.20146-03i	0.61731-04 -0.70652-04i	-0.98736-03 -0.48124-04i
300	-0.15993-02 -0.25589-02i	-0.20613-03 0.16446-03i	0.47605-04 -0.46937-04i	-0.64317-03 -0.30120-04i
400	-0.10519-02 -0.21036-02i	-0.20937-03 0.12043-03i	0.40930-04 -0.35101-04i	-0.44643-03 -0.21316-04i
500	-0.66275-03 -0.17227-02i	-0.19561-03 0.91319-04i	0.36128-04 -0.24936-04i	-0.34138-03 -0.12695-04i
600	-0.42634-03 -0.14509-02i	-0.17814-03 0.62275-04i	0.32641-04 -0.18134-04i	-0.28312-03 -0.83219-05i

TABLE B.1 (Continued)

Reduced Frequency = 3.25 Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.26015-01 -0.36405-01i	-0.10427-01 0.11736-01i	0.63763-03 -0.74126-03i	-0.14269-00 -0.76413-02i
300	-0.17139-01 -0.27417-01i	-0.11932-01 0.95178-02i	0.49134-03 -0.49376-03i	-0.93947-01 -0.48316-02i
400	-0.11272-01 -0.22538-01i	-0.12174-01 0.73459-02i	0.42607-03 -0.37094-03i	-0.67390-01 -0.31778-02i
500	-0.71022-02 -0.18547-01i	-0.10978-01 0.48755-02i	0.36982-03 -0.26139-03i	-0.48736-01 -0.19210-02i
600	-0.45017-02 -0.15545-01i	-0.98746-02 0.35473-02i	0.32337-03 -0.18561-03i	-0.40612-01 -0.13147-02i

Reduced Frequency = 3.50 Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.23126-02 -0.32360-02i	0.67336-03 0.74238-03i	0.40844-04 -0.45175-04i	-0.20347-02 -0.94639-04i
300	-0.15235-02 -0.24373-02i	0.75168-03 0.59233-03i	0.31643-04 -0.30746-04i	-0.13657-02 -0.59351-04i
400	-0.10020-02 -0.20034-02i	0.76814-03 0.45117-03i	0.27956-04 -0.22840-04i	0.91783-03 -0.38114-04i
500	-0.63133-03 -0.16407-02i	0.70842-03 0.31247-03i	0.24263-04 -0.15881-04i	-0.69317-03 -0.24137-04i
600	-0.40707-03 -0.13818-02i	0.64736-03 0.22641-03i	0.21654-04 -0.11367-04i	-0.63119-03 =0.15204-04i

TABLE B.1 (Continued)

Reduced Frequency = 4.00   Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.31796-03 -0.44495-03i	0.87694-04 0.98136-04i	0.13568-04 -0.15633-04i	-0.19364-04 -0.10148-05i
300	-0.20947-03 -0.33509-03i	0.95768-04 0.79139-04i	0.98837-05 -0.10364-04i	-0.13412-04 -0.60941-06i
400	-0.13768-03 -0.27547-03i	0.97367-04 0.58971-04i	0.89134-05 -0.78337-05i	-0.98716-05 -0.41368-06i
500	-0.86800-04 -0.22792-03i	0.91473-04 0.41788-04i	0.82155-05 -0.53214-05i	-0.68445-05 -0.26198-06i
600	-0.55963-04 -0.18999-03i	0.83212-04 0.29147-04i	0.68493-05 -0.41369-05i	-0.60310-05 -0.16217-06i

Reduced Frequency = 10.00   Mechanical Admittance Magnitude M = 0.2

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.11560-03 -0.16180-03i	0.41378-04 0.45113-04i	0.68314-05 -0.75634-05i	-0.64397-05 -0.33649-06i
300	-0.76165-04 -0.12185-03i	0.45367-04 0.35942-04i	0.48813-05 -0.50158-05i	-0.43164-05 -0.20938-06i
400	-0.50099-04 -0.10017-03i	0.46229-04 0.27148-04i	0.45137-05 -0.38426-05i	-0.27931-05 -0.13597-06i
500	-0.31565-04 -0.82039-04i	0.43237-04 0.18762-04i	0.42368-05 -0.25993-05i	-0.23102-05 -0.84910-07i
600	-0.20014-04 -0.69093-04i	0.40276-04 0.13647-04i	0.35362-05 -0.19338-05i	-0.17604-05 -0.64372-07i

TABLE B.1 (Continued)

Reduced Frequency = 0.02 Mechanical Admittance Magnitude M = 0.5

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	0.46240-01 0.64724-01i	0.42617-03 0.38126-03i	-0.38573-05 0.22647-05i	0.98437-04 0.12642-04i
300	0.30472-01 0.48737-01i	0.25195-03 0.50623-03i	-0.25679-05 0.13264-05i	0.11869-03 0.31645-04i
400	0.20021-01 0.40064-01i	0.15768-03 0.50162-03i	-0.22364-05 0.80126-06i	0.12439-03 0.32243-04i
500	0.12623-01 0.32815-01i	0.11472-03 0.49294-03i	-0.19714-05 0.55273-06i	0.12314-03 0.28136-04i
600	0.81382-02 0.27637-01i	0.92237-04 0.48410-03i	-0.17264-05 0.32419-06i	0.11629-03 0.25364-04i

Reduced Frequency = 0.025 Mechanical Admittance Magnitude M = 0.5

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.18207-00 0.25483-00i	0.76437-02 0.64169-02i	-0.73641-02 -0.41665-02i	0.34852-02 0.44219-03i
300	-0.11998-00 0.19189-00i	0.45331-02 -0.85720-02i	-0.50175-02 -0.22943-02i	0.41337-02 -0.11039-02i
400	-0.78836-01 0.15774-00i	0.31449-02 -0.84935-02i	-0.44320-02 -0.14362-02i	0.43621-02 -0.11845-02i
500	-0.49705-01 0.12920-00i	0.20324-02 -0.82736-02i	-0.37103-02 -0.10638-02i	0.42932-02 -0.98361-03i
600	-0.32044-01 0.10881-00i	0.16362-02 -0.80213-02i	-0.31639-02 -0.65741-02i	0.41436-02 -0.89445-03i

TABLE B.1 (Continued)

Reduced Frequency = 0.04 Mechanical Admittance Magnitude M = 0.5

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.69260-03	0.36943-03	-0.19873-04	0.12633-03
	0.97080-03i	0.32149-03i	-0.11570-04i	0.14362-04i
300	-0.45708-03	0.22358-03	-0.13572-04	0.14035-03
	0.73101-03i	-0.43627-03i	-0.62736-05i	-0.35921-04i
400	-0.30033-03	0.14935-03	-0.12014-04	0.15367-03
	0.60093-03i	-0.42886-03i	-0.38975-05i	-0.36575-04i
500	-0.18935-03	0.10643-03	-0.96427-05	0.15144-03
	0.49220-03i	-0.41045-03i	-0.28413-05i	-0.32140-04i
600	-0.12207-03	0.62374-04	-0.87510-05	0.14132-03
	0.41453-03i	-0.40310-03i	-0.20078-05i	-0.29436-04i

Reduced Frequency = 0.06 Mechanical Admittance Magnitude M = 0.5

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.46240-03	0.28374-03	-0.87253-05	0.10132-03
	0.64720-03i	-0.23631-03i	-0.49362-05i	-0.12289-04i
300	-0.30472-03	0.17295-03	-0.60139-05	0.11963-03
	0.48734-03i	-0.31268-03i	-0.30227-05i	-0.30417-04i
400	-0.20022-03	0.11296-03	-0.52461-05	0.12483-03
	0.40062-03i	-0.30847-03i	-0.17360-05i	-0.31249-04i
500	-0.12623-03	0.83259-04	0.44219-05	0.12361-03
	0.32813-03i	-0.29654-03i	-0.12642-05i	-0.28436-04i
600	-0.81382-04	0.62368-04	0.39265-05	0.11845-03
	0.27635-03i	-0.28232-03i	-0.89921-06i	-0.25147-04i

TABLE B.1 (Continued)

Reduced Frequency = 0.08 Mechanical Admittance Magnitude M = 0.5

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.80920-03 -0.11326-02i	0.33678-03 0.29620-03i	0.98425-05 -0.62194-05i	0.14236-03 0.16462-04i
300	-0.53313-03 -0.85285-03i	0.20147-03 0.40219-03i	0.66553-05 -0.34632-05i	0.16843-03 0.40687-04i
400	-0.35030-03 -0.70107-03i	0.12645-03 0.39642-03i	0.59142-05 -0.21258-05i	0.18269-03 0.41673-04i
500	-0.22091-03 -0.57422-03i	0.83629-04 0.38126-03i	0.49137-05 -0.15654-05i	0.17891-03 0.37780-04i
600	-0.14242-03 -0.48362-03i	0.62241-04 0.36764-03i	0.43177-05 -0.11267-05i	0.40571-03 0.33438-04i

Reduced Frequency = 0.10 Mechanical Admittance Magnitude M = 0.5

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.60690-01 0.84945-01i	-0.37912-01 0.32551-01i	0.22014-02 -0.12643-02i	0.60377-01 -0.72639-02i
300	-0.39988-01 -0.63964-01i	0.24622-01 0.42659-01i	0.14731-02 -0.73948-03i	0.73614-01 0.18445-01i
400	-0.26279-01 -0.52581-01i	0.14362-01 0.41238-01i	0.13240-02 -0.43655-03i	0.75267-01 0.18927-01i
500	-0.16568-01 -0.43067-01i	0.96357-02 0.39645-01i	0.11462-02 -0.31576-03i	0.74632-01 0.15945-01i
600	-0.10681-01 -0.36271-01i	0.71430-02 0.37845-01i	0.99873-03 -0.21437-03i	0.73412-01 -0.14501-01i

TABLE B.1 (Continued)

Reduced Frequency = 0.12 Mechanical Admittance Magnitude M = 0.5

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.68377-03 0.96473-03i	0.39213-03 0.34621-03i	0.22147-04 -0.12563-04i	0.59334-03 -0.71462-04i
300	-0.45060-03 -0.72644-03i	0.24651-03 0.45972-03i	0.14639-04 -0.70450-05i	0.72365-03 0.17955-03i
400	-0.29564-03 -0.59715-03i	0.16672-03 0.44867-03i	0.13562-04 -0.41651-05i	0.75121-03 0.18366-03i
500	-0.18667-03 -0.48912-03i	0.10236-03 0.43659-03i	0.11072-04 -0.30651-05i	0.74041-03 0.15614-03i
600	-0.12034-03 -0.41194-03i	0.86517-04 0.41360-03i	0.99362-05 -0.22641-05i	0.72109-03 -0.14836-03i

Reduced Frequency = 0.16 Mechanical Admittance Magnitude M = 0.5

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.26010-03 0.36405-03i	-0.21635-03 0.20146-03i	0.52639-05 0.30120-05i	0.85133-04 0.10372-04i
300	-0.17141-03 0.27413-03i	-0.13218-03 0.28462-03i	0.41362-05 0.17369-05i	0.10237-03 0.26130-04i
400	-0.11262-03 0.22535-03i	-0.86342-04 0.27659-03i	0.32490-05 0.11225-05i	0.10784-03 0.26845-04i
500	-0.71007-04 0.18547-03i	-0.62569-04 0.26836-03i	0.26349-05 0.75631-06i	0.10625-03 0.22643-04i
600	-0.45778-04 0.15545-03i	-0.45661-04 0.25167-03i	0.22614-05 0.52337-06i	0.10361-03 0.21043-04i

TABLE B.1 (Continued)

Reduced Frequency = 0.50 Mechanical Admittance Magnitude M = 0.5

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.12138-03 0.16989-03i	-0.98463-04 0.87912-04i	-0.10847-05 0.67312-06i	0.51446-04 0.62639-05i
300	-0.79989-04 -0.12793-03i	-0.57324-04 0.11436-03i	-0.69147-06 0.41362-06i	0.60395-04 0.15637-04i
400	-0.52557-04 -0.10516-03i	-0.30463-04 0.10592-03i	-0.60149-06 0.24632-06i	0.63774-04 0.15946-04i
500	-0.33137-04 -0.86134-04i	-0.23659-04 0.98746-04i	-0.51493-06 0.17241-06i	0.63142-04 0.13841-04i
600	-0.21363-04 -0.72543-04i	-0.17931-04 0.97240-04i	-0.46107-06 0.11579-06i	0.60226-04 0.12667-04i

Reduced Frequency = 0.70 Mechanical Admittance Magnitude M = 0.5

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.86704-04 0.12135-03i	-0.91778-04 0.80937-04i	0.10639-05 -0.64172-06i	0.55372-04 0.71073-05i
300	-0.57138-04 -0.91376-04i	-0.55935-04 0.10639-03i	0.71632-06 -0.35216-06i	0.66681-04 0.17743-04i
400	-0.37543-04 -0.75115-04i	-0.35662-04 0.10132-03i	0.62147-06 -0.23798-06i	0.68938-04 0.18013-04i
500	-0.23670-04 -0.61524-04i	-0.25184-04 0.94713-03i	0.53692-06 -0.16251-06i	0.67845-04 0.15936-04i
600	-0.15260-04 -0.51816-04i	-0.19627-04 0.89217-03i	0.45620-06 -0.10639-06i	0.65840-04 0.14324-04i

TABLE B.1 (Continued)

Reduced Frequency = 1.00 Mechanical Admittance Magnitude M = 0.5

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	0.17342-03 0.24270-03i	-0.11758-03 0.98436-04i	-0.15635-05 0.85637-06i	0.10442-03 0.12552-04i
300	0.11428-03 0.18275-03i	-0.68439-04 0.13046-03i	-0.11240-05 0.46731-06i	0.12837-03 0.30847-04i
400	0.75091-04 0.15231-03i	-0.42108-04 0.12914-03i	-0.90362-06 0.28476-06i	0.12684-03 0.31256-04i
500	0.47344-04 0.12305-03i	-0.30452-04 0.11378-03i	-0.79264-06 0.22361-06i	0.12479-03 0.28413-04i
600	0.30521-04 0.10363-03i	-0.24337-04 0.10225-03i	-0.62437-06 0.15235-06i	0.12126-03 0.25361-04i

Reduced Frequency = 1.20 Mechanical Admittance Magnitude M = 0.5

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.85891-03 -0.12556-02i	-0.11849-02 0.10364-02i	-0.70436-05 0.41698-05i	0.79684-01 -0.10362-01i
300	-0.56602-03 -0.94546-03i	-0.71475-03 0.13249-02i	-0.49175-05 0.23394-05i	0.96337-01 -0.25638-01i
400	-0.37195-03 -0.77722-03i	-0.45466-03 0.12647-02i	-0.42357-05 0.14162-05i	0.10194-00 -0.26238-01i
500	-0.23448-03 -0.63658-03i	-0.34102-03 0.11236-02i	-0.36113-05 0.10971-05i	0.99235-01 -0.22682-01i
600	-0.15117-03 -0.53614-03i	-0.23641-03 0.10475-02i	-0.30759-05 0.66488-06i	0.95838-01 -0.21034-01i

TABLE B.1 (Continued)

Reduced Frequency = 1.30 Mechanical Admittance Magnitude M = 0.5

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.11097-01 -0.15209-01i	-0.25163-01 0.21654-01i	-0.37521-04 0.21792-04i	0.45163-00 -0.51320-01i
300	-0.73129-02 -0.11452-01i	-0.15362-01 0.27946-01i	0.25273-04 0.12164-04i	0.54265-00 -0.12637-00i
400	-0.48050-02 -0.94143-02i	-0.82164-01 0.26751-01i	-0.21976-04 0.78434-05i	0.56377-00 -0.13149-00i
500	-0.30295-02 -0.77084-02i	-0.66370-01 0.25228-01i	-0.19123-04 0.53162-05i	0.55438-00 -0.11617-00i
600	-0.19531-02 -0.64942-02i	-0.43217-01 0.24103-01i	-0.17361-04 0.36413-05i	0.54175-00 -0.10436-00i

Reduced Frequency = 1.40 Mechanical Admittance Magnitude M = 0.5

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.11502-02 -0.16633-02i	0.15632-02 0.13643-02i	0.71352-05 -0.40371-05i	-0.79972-02 -0.10362-02i
300	-0.75798-03 -0.12525-02i	0.93625-03 0.17855-02i	0.47154-05 -0.23472-05i	-0.10365-01 -0.25372-02i
400	-0.49804-03 -0.10296-02i	0.52631-03 0.17453-02i	0.41934-05 -0.14361-05i	-0.10938-01 -0.26297-02i
500	-0.31400-03 -0.84329-03i	0.41639-03 0.16652-02i	0.35253-05 -0.10362-05i	-0.10744-01 -0.22648-02i
600	-0.20243-03 -0.71023-03i	0.32658-03 0.15297-02i	0.30567-05 -0.65343-06i	-0.10298-01i -0.20561-02i

TABLE B.1 (Continued)

Reduced Frequency = 2.00 Mechanical Admittance Magnitude M = 0.5

μ	H_1(ik)	H_2(ik)	T_1(ik)	T_2(ik)
200	-0.68207-04 -0.99917-04i	0.76394-04 0.65115-04i	0.78527-06 -0.46104-06i	-0.44839-04 -0.55178-05i
300	-0.44948-04 -0.75230-04i	0.45693-04 0.87212-04i	0.54632-06 -0.27149-06i	-0.53620-04 -0.13624-04i
400	-0.29534-04 -0.61941-04i	0.30975-04 0.85932-04i	0.47256-06 -0.16705-06i	-0.56217-04 -0.13918-04i
500	-0.18621-04 -0.50658-04i	0.20043-04 0.83269-04i	0.39620-06 -0.12127-06i	-0.55876-04 -0.11874-04i
600	-0.12004-04 -0.42664-04i	0.16228-04 0.81342-04i	0.34053-06 -0.78485-07i	-0.53321-04 -0.11014-04i

Reduced Frequency = 2.50 Mechanical Admittance Magnitude M = 0.5

μ	H_1(ik)	H_2(ik)	T_1(ik)	T_2(ik)
200	-0.52023-04 -0.70796-04i	0.60327-04 0.54639-04i	0.62819-06 -0.36154-06i	-0.40321-04 -0.50268-05i
300	-0.34283-04 -0.53309-04i	0.38241-04 0.70229-04i	0.43673-06 -0.20548-06i	-0.48136-04 -0.12634-04i
400	-0.22526-04 -0.43823-04i	0.23669-04 0.69134-04i	0.37285-06 -0.13781-06i	-0.49638-04 -0.12879-04i
500	-0.14202-04 -0.35894-04i	0.17276-04 0.67834-04i	0.31628-06 -0.98716-07i	-0.48936-04 -0.11657-04i
600	-0.91560-05 -0.30230-04i	0.13173-04 0.66215-04i	0.27334-06 -0.62048-07i	-0.47948-04 -0.10062-04i

TABLE B.1 (Continued)

Reduced Frequency = 0.02 Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$.
200	0.79856-02 0.11393-01i	0.39492-02 0.31030-02i	-0.14691-02 0.16518-02i	0.55262-03 0.11958-03i
300	0.78207-02 0.12946-01i	0.23731-02 0.17433-02i	-0.11470-02 0.10955-02i	0.35081-03 0.69724-04i
400	0.72144-02 0.14383-01i	0.15113-02 0.10741-02i	-0.99369-03 0.77842-03i	0.24693-03 0.44783-04i
500	0.62281-02 0.15719-01i	0.98072-03 0.64177-03i	-0.90230-03 0.55989-03i	0.18376-03 0.27978-04i
600	0.48859-02 0.16886-01i	0.63819-03 0.31470-03i	-0.83550-03 0.39081-03i	0.14185-03 0.14532-04i

Reduced Frequency = 0.04 Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.17358-02 0.41435-02i	0.72182-02 0.15006-02i	-0.12636-03 -0.12202-03i	0.75615-03 0.84050-04i
300	-0.19740-02 0.43312-02i	0.54771-02 -0.21233-03i	-0.10068-03 -0.11066-03i	0.52024-03 -0.59133-05i
400	-0.24861-02 0.41541-02i	0.44567-02 -0.94187-03i	-0.72640-04 -0.12230-03i	0.39814-03 -0.45829-04i
500	-0.29947-02 0.36643-02i	0.38788-02 -0.12356-02i	-0.41473-04 -0.13096-03i	0.32910-03 -0.64133-04i
600	-0.33110-02 0.29925-02	0.35198-02 -0.12894-02i	-0.11717-04 -0.13011-03i	0.28649-03 -0.69816-04i

TABLE B.1 (Continued)

Reduced Frequency = 0.06 Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.42285-02 0.68309-02i	0.71873-02 -0.18318-02i	-0.35204-03 -0.28100-03i	0.34122-03 -0.41386-04i
300	-0.57434-02 0.47592-02i	0.98000-02 -0.18067-02i	-0.12573-03 -0.28425-03i	0.40299-03 -0.10222-03i
400	-0.56964-02 0.30550-02i	0.10195-01 -0.14454-02i	-0.18915-04 -0.22210-03i	0.43523-03 -0.10445-03i
500	-0.51404-02 0.19944-02i	0.95330-02 -0.12143-02i	-0.25673-04 -0.16561-03i	0.42802-03 -0.94459-04i
600	-0.45284-02 0.13606-02i	0.86301-02 -0.10887-02i	0.43258-04 -0.12478-03i	0.40311-03 -0.84556-04i

Reduced Frequency = 0.08 Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.93049-02 -0.34686-04i	0.16153-01 0.13645-01i	0.11725-04 -0.46341-03i	0.10992-02 0.52227-03i
300	-0.51556-02 -0.26764-02i	0.93289-02 0.17871-01i	0.12633-03 -0.18458-03i	0.71634-03 0.75880-03i
400	-0.30796-02 -0.24674-02i	0.59206-02 0.17459-01i	0.10067-03 -0.84026-04i	0.49660-03 0.76480-03i
500	-0.20999-02 -0.20600-02i	0.42942-02 0.16773-01i	0.76617-04 -0.45342-04i	0.37848-03 0.74290-03i
600	-0.15631-02 -0.17295-02i	0.33848-02 0.16221-01i	0.60289-04 -0.27403-04	0.30657-03 0.72185-03i

TABLE B.1 (Continued)

Reduced Frequency = 0.10   Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.23296-02 0.10879-03i	0.73786-02 0.80919-02i	0.13804-03 -0.76615-04i	0.13477-02 -0.71010-04i
300	-0.20256-02 -0.75452-04i	0.81458-02 0.64850-02i	0.10211-03 -0.42908-04i	0.11450-02 0.43424-04i
400	-0.17842-02 -0.86613-04i	0.81471-02 0.47728-02i	0.80940-04 -0.26903-04i	0.10089-02 0.39569-04i
500	-0.15787-02 -0.70848-04i	0.76904-02 0.33812-02i	0.67115-04 -0.18091-04i	0.89347-03 0.11318-04i
600	-0.14036-02 -0.39685-04i	0.70612-02 0.23653-02i	0.57404-04 -0.12800-04i	0.79328-03 -0.14842-04i

Reduced Frequency = 0.12   Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.18772-02 0.10661-02i	-0.90329-03 0.52585-02i	0.81006-04 -0.32235-04i	0.25395-03 0.89444-04i
300	-0.15513-02 0.69970-03i	-0.18749-02 0.42784-02i	0.70328-04 -0.14844-04i	0.10844-03 0.68126-04i
400	-0.13303-02 0.48858-03i	-0.22072-02 0.34084-02i	0.60420-04 -0.71915-05i	0.47250-04 0.44167-04i
500	-0.11622-02 0.35231-03i	-0.22639-02 0.27391-02i	0.52525-04 -0.30475-05i	0.18610-04 0.26347-04i
600	-0.10284-02 0.26019-03i	-0.22038-02 0.22374-02i	0.46245-04 -0.59567-06i	0.43124-05 0.14059-04i

TABLE B.1 (Continued)

Reduced Frequency = 0.16 Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.10851-02 0.67026-03i	-0.65192-03 0.28731-02i	0.45289-04 0.10332-04i	0.29555-03 0.15432-04i
300	-0.90257-03 0.42788-03i	-0.98139-03 0.20709-02i	0.34265-04 0.10339-04i	0.18479-03 0.90165-05i
400	-0.76263-03 0.28982-03i	-0.10122-02 0.15461-02i	0.27459-04 0.96001-05i	0.12954-03 0.58165-05i
500	-06.5467-03 0.20643-03i	-0.94396-03 0.12029-02i	0.22837-04 0.87955-05i	0.96650-04 0.38670-05i
600	-0.57068-03 0.15327-03i	-0.88425-03 0.96958-03i	0.19504-04 0.80482-05i	0.74031-04 0.25710-05i

Reduced Frequency = 0.20 Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.70618-03 0.39414-03i	-0.14051-02 0.93660-3i	0.24696-04 0.19117-04i	0.19572-03 0.89543-05i
300	-0.54124-03 0.20962-03i	-0.10824-02 0.56307-03i	0.20621-04 0.15615-04i	0.12232-03 0.52230-05i
400	-0.43026-03 0.12994-03i	-0.86694-03 0.38555-03i	0.16891-04 0.13207-04i	0.85630-04 0.33593-05i
500	-0.35481-03 0.88942-04i	-0.71922-03 0.28691-03i	0.14131-04 0.11392-04i	0.63610-04 0.22385-05i
600	-0.30107-03 0.65134-04i	-0.61312-03 0.22572-03i	0.12089-04 0.99882-05i	0.48930-04 0.14923-05

TABLE B.1 (Continued)

Reduced Frequency = 0.50 Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.10192-03 0.18915-05i	-0.24138-03 0.13861-03i	-0.32162-05 0.50880-05i	0.85685-04 0.52259-05i
300	-0.68039-04 -0.31067-05i	-0.16778-03 0.83710-04i	-0.21969-05 0.36082-05i	0.53088-04 0.23681-05i
400	-0.50906-04 -0.39600-05i	-0.12808-03 0.59281-04i	-0.16755-05 0.27841-05i	0.37559-04 0.19525-05i
500	-0.40627-04 -0.39438-05i	-0.10345-03 0.45694-04i	-0.13556-05 0.22637-05i	0.28037-04 0.13289-05i
600	-0.33788-04 -0.37144-05i	-0.86727-04 0.37100-04i	-0.11388-05 0.19063-05i	0.21568-04 0.88818-05i

Reduced Frequency = 0.90 Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.45535-04 0.93735-06i	-0.77013-04 0.20722-03i	0.66737-06 -0.13689-05i	0.60316-04 0.32612-05i
300	-0.29952-04 -0.29319-06i	-0.56665-04 0.13378-03i	0.40545-06 -0.87057-06i	0.37769-04 0.19114-05i
400	-0.22260-04 -0.55784-06i	-0.44458-04 0.98321-04i	0.28758-06 -0.63918-06i	0.26563-04 0.12784-05i
500	-0.17697-04 -0.60566-06i	-0.36489-04 0.77597-04i	0.22175-06 -0.50531-06i	0.19837-04 0.91510-06i
600	-0.14681-04 -0.59206-06i	-0.30912-04 0.64043-04i	0.18004-06 -0.41794-06i	0.15079-04 0.58833-06i

TABLE B.1 (Continued)

Reduced Frequency = 0.98 Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.26769-04 -0.85188-05i	-0.67352-04 0.21114-03i	-0.19635-05 0.11527-05i	-0.31220-04 -0.16099-02i
300	-0.17010-04 -0.61722-05i	-0.48032-04 0.13346-03i	-0.13505-05 0.77467-06i	0.11395-04 -0.10304-02i
400	-0.12451-04 -0.47936-05i	-0.37049-04 0.97448-04i	-0.10279-05 0.58266-06i	0.19553-04 -0.75713-03i
500	-0.98151-05 -0.39089-05i	-0.30095-04 0.76711-04i	-0.82943-06 0.46677-06i	0.20667-04 -0.59827-03i
600	-0.80992-05 -0.32969-05i	-0.25321-04 0.63242-04i	-0.69508-06 0.38928-06i	0.19928-04 -0.49446-03i

Reduced Frequency = 1.00 Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	0.23274-02 0.83762-02i	-0.49514-02 0.63041-01i	-0.27985-04 0.12472-04i	0.65219-00 0.56613-00i
300	0.10921-02 0.36632-02i	-0.19464-02 0.41722-01i	-0.11183-04 0.89467-05i	0.41736-00 0.43271-00i
400	0.62282-03 0.20396-02i	-0.10303-02 0.31036-01i	-0.58346-05 0.58528-05i	0.29174-00 0.21856-00i
500	0.40120-03 0.21975-02i	-0.63796-03 0.24674-01i	-0.35132-05 0.40104-05i	0.21651-00 0.14233-00i
600	0.28001-03 0.89789-03i	-0.43482-03 0.20465-01i	-0.23111-05 0.28828-05i	0.16256-00 0.99056-01i

TABLE B.1 (Continued)

Reduced Frequency = 1.02   Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.23894-04 -0.10076-04i	-0.60252-04 0.19517-03i	-0.17824-05 0.15345-05i	-0.28668-04 -0.14991-02i
300	-0.15167-04 -0.70986-05i	-0.42758-04 0.12335-03i	-0.12266-05 0.10246-05i	0.84268-05 -0.95861-03i
400	-0.11097-04 -0.54506-05i	-0.32918-04 0.90074-04i	-0.93381-06 0.76857-06i	0.15748-04 -0.70408-03i
500	-0.87457-05 -0.44174-05i	-0.26713-04 0.70911-04i	-0.75356-06 0.61479-06i	0.16906-04 -0.55623-03i
600	-0.72157-05 -0.37115-05i	-0.22461-04 0.58462-04i	-0.63153-06 0.51225-06	0.16409-04 -0.45966-03i

Reduced Frequency = 1.10   Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.34650-03 -0.59857-05i	-0.52784-04 0.29375-03i	0.23250-05 -0.12770-06i	-0.50639-04 -0.25669-05i
300	-0.22210-04 -0.42847-05i	-0.38463-04 0.18105-03i	0.15104-05 -0.99077-07i	-0.31743-04 -0.15174-05i
400	-0.16349-04 -0.33148-05i	-0.29938-04 0.31074-03i	0.11189-05 -0.78953-07i	-0.22631-05 -0.99674-06i
500	-0.12936-04 -0.26982-05i	-0.24442-04 0.10227-03i	0.88864-06 -0.65264-07i	-0.17436-04 -0.72613-06i
600	-0.10702-04 -0.22735-05i	-0.20630-04 0.83979-04i	0.73701-06 -0.55514-07i	-0.12459-04 -0.40557-06i

TABLE B.1 (Continued)

Reduced Frequency = 1.50 Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.17556-04 -0.48599-05i	0.16482-03 0.45628-03i	0.49043-06 -0.22569-05i	-0.35726-04 -0.23157-05i
300	-0.104672-04 -0.31937-05i	0.80551-04 0.26402-03i	0.29199-06 -0.14516-05i	-0.22611-04 -0.14556-05i
400	-0.74787-05 -0.23668-05i	0.52141-04 0.18556-03i	0.20928-06 -0.10727-05i	-0.16378-04 -0.98635-06i
500	-0.58227-05 -0.18784-05i	0.38253-04 0.14301-03i	0.16339-06 -0.85132-06i	-0.12872-04 -0.61467-06i
600	-0.47688-05 -0.15567-05i	0.30107-04 0.11633-03i	0.13410-06 -0.70587-06i	-0.98889-05 -0.42146-06i

Reduced Frequency = 1.60 Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.89327-05 -0.30113-05i	0.16532-04 0.68915-04i	0.62887-05 -0.41736-05i	-0.33829-04 -0.21616-05i
300	-0.62165-04 -0.19287-05i	0.10278-03 0.43882-04i	0.37168-05 -0.25322-05i	-0.21364-05 -0.12373-05i
400	-0.44472-05 -0.14893-05i	0.75617-05 0.30887-04i	0.25417-05 -0.19663-05i	-0.14472-04 -0.74479-06i
500	-0.37614-05 -0.10836-05i	0.49211-05 0.22314-05i	0.17746-05 -0.14103-05i	-0.10436-04 -0.58225-06i
600	-0.31152-05 -0.87316-06i	0.34364-05 0.17225-04i	0.12564-05 -0.92747-06i	-0.83456-05 -0.30262-06i

TABLE B.i (Continued)

Reduced Frequency = 1.67 Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.84668-05 -0.28446-05i	0.16147-04 0.66839-04i	0.61174-05 -0.40236-05i	-0.33226-04 -0.19987-05i
300	-0.60812-05 -0.18981-05i	0.99746-05 0.43161-04i	0.36814-05 -0.24132-05i	-0.21748-04 -0.12126-05i
400	-0.42864-05 -0.14175-05i	0.74287-05 0.30336-04i	0.24278-05 -0.19113-05i	-0.16573-04 -0.78451-06i
500	-0.36214-05 -0.10366-05i	0.48007-05 0.22061-04i	0.17542-05 -0.31905-05i	-0.11768-04 -0.54362-06i
600	-0.29414-05 -0.84167-06i	0.32874-05 0.16932-04i	0.12217-05 -0.91286-06i	-0.95359-05 -0.34231-06i

Reduced Frequency = 2.00 Mechanical Admittance Magnitude M = 0.7

μ	$H_1(ik)$	$H_2(ik)$	$T_1(ik)$	$T_2(ik)$
200	-0.86315-05 -0.25004-05i	0.16224-04 0.57218-04i	0.12568-06 0.69552-06i	-0.30172-04 -0.18248-05i
300	-0.51647-05 -0.19237-05i	0.10117-04 0.36238-04i	0.81715-07 0.46158-06i	-0.18775-04 -0.10823-05i
400	-0.39713-05 -0.13772-05i	0.81793-05 0.27184-05i	0.56236-07 0.36143-06i	-0.14827-04 -0.68338-06i
500	-0.29117-05 -0.10216-05i	0.63177-05 0.19735-04i	0.43756-07 0.28552-06i	-0.99754-05 -0.42663-06i
600	-0.22188-05 -0.86504-06i	0.52119-05 0.14823-04i	0.35713-07 0.23462-06i	-0.74611-05 -0.29793-06i

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